# SOLID GEOMETRY

AND

# SPHERICAL TRIGONOMETRY

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# PREFACE

In constructing this book the author has endeavored to combine the freedom, scope and informality of modern methods with the substantial qualities characteristic of those "old reliable" textbooks which have stood the test of time and everyday use. Every effort has been made to develop the text with clarity, precision and a reasonable degree of rigor.

Many of the theorem proofs have an algebraic form and flavor, and occasionally simple trigonometric relations are employed in order to achieve results gracefully. The text is accompanied by a genuine abundance of exercises of all types and of varying degrees of difficulty, including numerous exercises involving geometric proofs as well as those requiring computations. These exercises commence at once, the earlier ones serving to induce the student to think in three dimensions at the start without being conscious of tackling a task too utterly new to him.

The arrangement of the entire book is such as to provide ready reference. There are eighteen chapters, each one being the exposition of one principal idea; and each important item — whether or not it constitutes a complete paragraph or section — bears a number (§). The author is aware of the importance of flexibility in the use of the book, and therefore has sought to provide a textbook which is adaptable not only to a full course in the subject but also to a condensed survey. "Solid Geometry Made Easy" is distinctly not the motif. Nevertheless, all material is presented with the intention of bringing it within sure reach of the student who is willing to read carefully and thoughtfully and follow directions with honest effort and perseverance.

The four concluding chapters (15–18) constitute an introductory course in Spherical Trigonometry with applications. Basic theorems are developed rigorously as a natural and logical extension of the spherical geometry studied at the end of the Solid Geometry section of the book. Here again a maximum of flexibility is provided. For those who require merely a knowledge of spherical right triangle solution by Napier's Rule Chapter 15 is sufficient. For those who demand a medium length survey with a few interesting applications Chapters 15, 16 and selected portions of Chapter 17 may be covered. Chapter 18 contains supplementary formulas for solving spherical oblique triangles and may be taken or omitted at will. If it is thought desirable the formulas of Chapter 18 may be acquired before the applications of Chapter 17 are discussed, although the latter chapter does not require a knowledge of Chapter 18. For the study

of Spherical Trigonometry a knowledge of Plane Trigonometry is of course a prerequisite.

This book is a revision and expansion of the author's earlier text (privately lithoprinted) which has been used for the past three years in the Phillips Exeter Academy mathematics department. All changes, omissions and additions have been made with the intention not only of correcting the faults of the preprint edition but also of meeting more satisfactorily the present day demands in secondary school mathematics. To his colleagues the author makes grateful acknowledgment of their criticisms and suggestions which have been of great assistance to him in the preparation of this textbook.

Henry L. C. Leighton

Exeter, N. H.
June, 1943

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## Reference Lists

### Terms Used in Plane Geometry

*Acute Angle:* An angle which is less than 90°.

*Adjacent Angles:* Angles in the same plane which have a common vertex and a common side between them.

*Altitude of Parallelogram:* The perpendicular distance between either pair of opposite sides.

*Altitude of Trapezoid:* The perpendicular distance between the bases.

*Altitude of Triangle:* The perpendicular distance from a vertex to the opposite side.

*Angles formed by a Transversal* (Fig. 1):

Alternate-interior angles: $e$ and $c$, or $f$ and $d$.

Alternate-exterior angles: $h$ and $b$, or $g$ and $a$.

$\left.\begin{array}{l}\text{Corresponding angles}\\ \text{Exterior-interior angles}\end{array}\right\}$: $g$ and $c$, $h$ and $d$, $f$ and $b$, $e$ and $a$.

Co-interior angles: $f$ and $c$, or $e$ and $d$.

Co-exterior angles: $g$ and $b$, or $h$ and $a$.

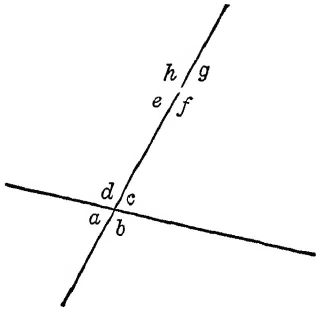



Fig. 1

*Apothem of Regular Polygon:* The radius of the inscribed circle, or the perpendicular distance from the center to any side.

*Arc of Circle:* A portion of the circumference.

*Bases of Trapezoid:* The parallel sides of the trapezoid.

*Center of Gravity of Triangle:* The intersection of the medians.

*Center of Parallelogram:* The intersection of the diagonals.

*Center of Regular Polygon:* The point which is the center both of the inscribed and circumscribed circles.

*Central Angle of a Circle:* An angle with vertex at the center of the circle and with a radius for each side.

*Central Angle of a Regular Polygon:* An angle with vertex at the center of the polygon, with a radius of the polygon for each side, and subtending a side of the polygon.

*Centroid of Triangle:* The intersection of the medians. (See Center of Gravity above.)

*Chord of Circle:* A straight line terminated by two points on the circumference.

*Circumcenter of Triangle:* The center of the circumscribed circle.

*Circumscribed Polygon:* A polygon having each side tangent to a given circle. (Here the circle is inscribed in the polygon.)

*Complements:* Angles whose sum is 90° (usually two angles).

*Concentric Circles:* Circles having a common center.

*Concurrent Lines:* Lines passing through a common point.
*Concyclic Points:* Points which lie on the circumference of a circle.
*Congruent Figures:* Figures which can be made to coincide.
*Conjugates:* Angles whose sum is 360°.
*Convex Polygon:* A polygon each angle of which is less than 180°. (A polygon is concave if it is not convex.)
*Cyclic Polygon:* A polygon with all vertices lying on the circumference of a circle.
*Degree:* An angle which is $\frac{1}{90}$ of a right angle.
*Degree of Arc:* An arc of a circle which subtends a central angle of 1°. A degree of arc is $\frac{1}{360}$ of any circumference.
*Exterior Angle of Polygon:* An angle formed by one side and the extension of an adjacent side.
*External Division of Line-segment:* If $BP$ is an extension of a line-segment $AB$, $P$ is said to divide $AB$ externally into the segments $PA$ and $PB$.
*Hypotenuse:* The side opposite the right angle in a right triangle.
*Incenter of Triangle:* The center of the inscribed circle.
*Inscribed Angle in a Circle:* An angle with vertex on the circumference and having a chord for each of its sides.
*Inscribed Polygon:* A polygon with its vertices on the circumference of a circle. (Here the circle is circumscribed about the polygon.)
*Isosceles Trapezoid:* A trapezoid whose legs are equal.
*Isosceles Triangle:* A triangle having two equal sides.
*Legs of an Isosceles Triangle:* The two equal sides of the triangle.
*Legs of a Right Triangle:* The sides including the right angle.
*Legs of a Trapezoid:* The two non-parallel sides of the trapezoid.
*Limit:* If the value of a variable quantity $x$ approaches the value of a certain constant $k$ in such a way that the difference between $k$ and $x$ becomes and remains less than any preassigned quantity, however small, then $x$ is said to approach $k$ as a limit.
*Locus:* The path traced by a point which moves in accordance with specified geometric conditions. A locus is the place where there are all possible points which satisfy certain given geometric conditions.
*Major Arc of a Circle:* An arc which is greater than a semicircle.
*Mean proportional:* $m$ is a mean proportional between $a$ and $b$ if $m^2 = a \cdot b$.
*Median of Trapezoid:* The line connecting the mid-points of the two legs.
*Median of Triangle:* A line connecting a vertex with the mid-point of the opposite side.
*Minor Arc of a Circle:* An arc which is less than a semicircle.
*Obtuse Angle:* An angle which is greater than 90° and less than 180°.
*Orthocenter of a Triangle:* The intersection of the altitudes.
*Parallelogram:* A quadrilateral with the opposite sides parallel.
*Parallel Lines:* Lines in the same plane which can never meet however far they may be produced.
*Perigon:* The total angular space about a point in a plane, that is, 360°.

*Projection of a Line-segment upon a Line:* If from the extremities of an external line-segment $s$ perpendiculars are drawn to a given line $m$, meeting $m$ at $A$ and $B$, respectively, the segment $AB$ is the projection of $s$ upon $m$.

*Projection of a Point upon a Line:* If from an external point $P$ a line is drawn perpendicular to a given line $m$, meeting $m$ at $A$, point $A$ is the projection of $P$ upon the line $m$.

*Radius of a Regular Polygon:* The radius of the circumscribed circle, or, the distance from the center of the polygon to any vertex.

*Rectangle:* A parallelogram with each angle a right angle.

*Reflex Angle:* An angle which is greater than 180°.

*Regular Polygon:* A polygon having equal sides and equal angles.

*Rhomboid:* A parallelogram whose adjacent sides are unequal and whose angles are not right angles.

*Rhombus:* A parallelogram whose four sides are equal and whose angles are not right angles.

*Right Angle:* When one straight line meets another straight line in such a way that the adjacent angles thus formed are equal, either of these angles is called a right angle. A right angle contains 90°. When one line meets another line at right angles, the two lines are said to be *perpendicular* to each other.

*Right and Left Sides of an Angle:* Imagine a person to be standing at the vertex of an angle and looking out between the sides of the angle. The side appearing at the right of the observer is the right side of the angle; the other side is the left side. In the angle $ABC$, $BC$ is the right side; $BA$ is the left side.

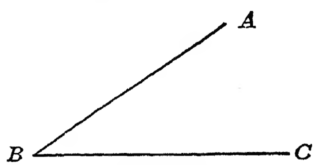


FIG. 2

*Right Triangle:* A triangle one of whose angles is a right angle.

*Secant of a Circle:* A line cutting the circumference in two points.

*Sector of a Circle:* The figure bounded by two radii and either of the arcs intercepted by those radii.

*Segment of a Circle:* The figure bounded by a chord and either of the arcs subtended by that chord.

*Semicircle:* One-half of a circle.

*Similar Polygons:* Polygons whose corresponding angles are equal and arranged in the same order, and whose corresponding sides are proportional and arranged in the same order.

*Square:* A parallelogram with four equal sides and four right angles.

*Straight Angle:* An angle whose sides extend in opposite directions so as to form a straight line. A straight angle contains 180°.

*Supplements:* Angles whose sum is 180° (usually two angles).

*Tangent Circles:* Circles in the same plane which are tangent to the same line at the same point.

*Tangent to a Circle:* A line lying in the same plane with a given circle and touching the circumference at one and only one point, no matter how far extended.

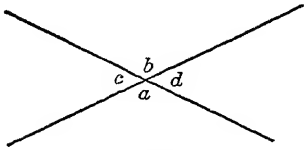



FIG. 3

*Transversal:* If a line $t$ cuts several lines $a$, $b$, $c$, etc., $t$ is called a transversal of the lines $a$, $b$, $c$, etc.

*Trapezoid:* A quadrilateral with two sides parallel and the other two sides not parallel.

*Trapezium:* A quadrilateral none of whose sides are equal and none of whose angles are equal.

*Vertical Angles:* The pairs of opposite angles formed when one straight line intersects another straight line. In the figure $a$ and $b$, or $c$ and $d$ are vertical angles.

## Plane Geometry Theorems

(The following theorems are not listed in logical sequence, but are arranged in classified groups for ready reference. Any polygons mentioned are convex polygons.)

I. *Congruent Triangles*

1. Two triangles are congruent if two sides and the included angle of one are respectively equal to two sides and the included angle of the other. $SAS = SAS$.
2. Two triangles are congruent if two angles and the included side of one are respectively equal to two angles and the included side of the other. $ASA = ASA$.
3. Two triangles are congruent if the three sides of one are respectively equal to the three sides of the other. $SSS = SSS$.
4. Two triangles are congruent if a side and any two angles of one are respectively equal to a side and two angles of the other. $AAS = AAS$.
5. Two right triangles are congruent if the hypotenuse and an acute angle of one are respectively equal to the hypotenuse and an acute angle of the other. $HA = HA$.
6. Two right triangles are congruent if the hypotenuse and a leg of one are respectively equal to the hypotenuse and leg of the other. $HL = HL$.

II. *Isosceles Triangles*

7. If two sides of a triangle are equal, the angles opposite these sides are equal.
   A. If a triangle is equilateral it is also equiangular.
   B. The bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base.
   C. The perpendicular bisector of the base of an isosceles triangle passes through the vertex.

D. In an isosceles triangle the median from the vertex is perpendicular to the base.

8. If two angles of a triangle are equal, the sides opposite these angles are equal.

   A. If a triangle is equiangular it is also equilateral.

III. *Right triangles*

9. The mid-point of the hypotenuse of a right triangle is equidistant from all three vertices.
10. If the acute angles of a right triangle are respectively 30° and 60°, the hypotenuse is twice the shorter leg.
11. If the hypotenuse of a right triangle is twice the shorter leg, the acute angles are respectively 30° and 60°.
12. In a right triangle if an altitude is drawn to the hypotenuse,
    i) the triangle is divided into two right triangles which are similar to the given triangle and similar to each other;
    ii) the altitude is the mean proportional between the projections of the legs upon the hypotenuse;
    iii) either leg is the mean proportional between the hypotenuse and the projection of that leg upon the hypotenuse.
13. In any right triangle the square of the hypotenuse equals the sum of the squares of the two legs.
14. If the square of the longest side of a triangle equals the sum of the squares of the other two sides, the triangle is a right triangle.

IV. *Triangles in General*

(a) Inequalities

15. If two sides of a triangle are unequal, the angles opposite these sides are unequal, and the angle opposite the greater side is the greater.

    A. In any triangle the angle opposite the greatest side is the greatest angle.

16. If two angles of a triangle are unequal, the sides opposite these angles are unequal, and the side opposite the greater angle is the greater.

    A. In any triangle the side opposite the greatest angle is the greatest side.

17. If two triangles have two sides of one equal to two sides of the other, but with the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.
18. If two triangles have two sides of one equal to two sides of the other, but with the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

(b) Concurrent Lines. Centers of a Triangle

19. The perpendicular bisectors of the sides of any triangle meet in a point which is equidistant from all three vertices. This point is the *circumcenter*.
20. The bisectors of the angles of any triangle meet in a point which is equidistant from all three sides. This point is the *incenter*.
21. The altitudes of a triangle meet in a point. This point is the *orthocenter*.
22. The medians of a triangle meet in a point which is two-thirds of the way from any vertex to the mid-point of the opposite side. This point is the *center of gravity* or *centroid*.

(c) Proportional Division

23. If a line cuts two sides of a triangle and is parallel to the third side, it divides the first two sides proportionally in the same sense.
24. If a line cuts two sides of a triangle so as to divide those two sides proportionally in the same sense, this line is parallel to the third side.
25. The bisector of an interior angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides of the triangle, and conversely.
26. The bisector of an exterior angle of a triangle divides the opposite side externally into segments which are proportional to the adjacent sides of the triangle, and conversely.

(d) Mid-points

27. If a line bisects one side of a triangle and is parallel to a second side, it bisects the third side.
28. If a line connects the mid-points of two sides of a triangle, it is parallel to the third side and equals one-half the third side.

(e) Angle Sums

29. The sum of the interior angles of any triangle is 180°.
    A. An exterior angle of a triangle equals the sum of the two remote interior angles.
    B. An exterior angle of a triangle is greater than either of the remote interior angles.
    C. A triangle cannot have more than one right angle or more than one obtuse angle.
    D. If two triangles have two angles of one respectively equal to two angles of the other, then the third angles are equal.

V. *Angle Sums in Polygons*

30. If a polygon has $n$ sides, the sum of the interior angles is $(n - 2)180°$.
    A. If a regular polygon has $n$ sides, each angle equals

$$\frac{(n-2)180°}{}$$

31. The sum of the exterior angles of a polygon made by extending each of the sides once in succession is 360°.

VI. *Parallels and Perpendiculars*

32. From an external point one and only one line can be drawn perpendicular to a given line.
33. The shortest distance from an external point to a given line is the perpendicular from that point to the line.
34. Two lines perpendicular to the same line are parallel, provided that all three lines lie in a specified plane.
35. If a line is perpendicular to one of two parallel lines, it is perpendicular to the other, also, provided that all three lines lie in a specified plane.
36. If two parallel lines are cut by a transversal,
    i) the alternate-interior angles are equal;
    ii) the interior-exterior (corresponding) angles are equal;
    iii) the alternate-exterior angles are equal;
    iv) the co-interior angles are supplementary;
    v) the co-exterior angles are supplementary.
37. If two lines in the same plane are cut by a transversal, they are parallel if
    i) the alternate-interior angles are equal;
    ii) the interior-exterior (corresponding) angles are equal;
    iii) the alternate-exterior angles are equal;
    iv) the co-interior angles are supplementary;
    v) the co-exterior angles are supplementary.
38. The perpendicular distance between two given parallel lines is constant.

43-A. If two parallel lines intersect two other parallel lines, either pair cuts off equal segments on the other pair.

39. If three or more parallel lines in the same plane have equal intercepts on one transversal, they have equal intercepts on any other transversal.
40. Three or more parallel lines in the same plane have proportional intercepts on any two transversals.
41. Two angles in the same plane are equal
    i) if their sides are parallel, right-to-right, left-to-left;
    ii) if their sides are respectively perpendicular, right-to-right, left-to-left.
42. Two angles in the same plane are supplementary
    i) if their sides are parallel, right-to-left, left-to-right;
    ii) if their sides are respectively perpendicular, right-to-left, left-to-right.

VII. *Parallelograms*

43. The opposite sides of a parallelogram are equal.
    A. If two parallel lines intersect two other parallel lines, either pair cuts off equal segments on the other pair.
    B. The adjacent angles of a parallelogram are supplementary; the opposite angles are equal.
44. If the opposite sides of a plane quadrilateral are equal, the figure is a parallelogram.
45. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.
46. The diagonals of a parallelogram bisect each other.
47. If the diagonals of a quadrilateral bisect each other the figure is a parallelogram.
48. The diagonals of a rhombus are perpendicular bisectors of each other, and they bisect the angles of the rhombus.
49. The diagonals of a rectangle are equal; the diagonals of a square are equal.
    A. The diagonals of a square are perpendicular bisectors of each other, and they bisect the angles of the square.

VIII. *Trapezoids*

50. The median of a trapezoid is parallel to both bases, and equals one-half the sum of the bases.
51. The base angles of an isosceles trapezoid are equal.
52. If the base angles of a trapezoid are equal, the trapezoid is isosceles.

IX. *Similar Triangles*

53. Two triangles are similar if two angles of one respectively equal two angles of the other.
    A. Two right triangles are similar if an acute angle of one equals an acute angle of the other.
54. Two triangles are similar if two sides of one are proportional to two sides of the other and the included angle of the first equals the included angle of the second.
    A. A line intersecting two sides of a triangle and parallel to the third side cuts off a triangle which is similar to the given triangle.
55. Two triangles are similar if the three sides of one are respectively proportional to the three sides of the other.
56. In similar triangles, corresponding altitudes and corresponding medians have the same ratio as any two corresponding sides.
57. The perimeters of two similar triangles have the same ratio as any two corresponding sides.
58. The areas of two similar triangles have the same ratio as
    i) the squares of any two corresponding sides;

ii) the squares of corresponding altitudes or medians;
iii) the squares of the perimeters.

X. *Similar Polygons*

59. If two polygons are similar they can be divided by corresponding diagonals into pairs of similar triangles which are similarly placed.
    A. If two polygons are similar, corresponding diagonals have the same ratio as any two corresponding sides and the same ratio as the perimeters.
60. If two polygons of the same number of sides can be divided by corresponding diagonals into pairs of triangles which are similar and similarly placed, the polygons are similar.
61. Two regular polygons of the same number of sides are similar.
    A. In two regular polygons of the same number of sides, the perimeters, apothems, or radii have the same ratio as any two corresponding sides.
62. The areas of two similar polygons have the same ratio as
    i) the squares of any two corresponding sides;
    ii) the squares of corresponding diagonals;
    iii) the squares of the perimeters.
    A. If two regular polygons have the same number of sides, their areas have the same ratio as
        i) the squares of any two corresponding sides;
        ii) the squares of their apothems;
        iii) the squares of their radii;
        iv) the squares of their perimeters.

XI. *Regular Polygons*

63. A circle can be circumscribed about or inscribed in any regular polygon.
    A. The radii of a regular polygon form equal angles at the center.
    B. The radii of a regular polygon bisect the interior angles.
    C. Any interior angle of a regular polygon is the supplement of any one of the central angles.
    D. Any apothem of a regular polygon bisects the side to which it is drawn.

XII. *Circles.* (Chords, tangents, secants)

64. A diameter perpendicular to a chord bisects that chord and both arcs subtended by that chord.
65. The perpendicular bisector of a chord passes through the center of the circle.
66. If two chords of a circle are equal, they are equidistant from the center of the circle, and conversely.
67. If two chords of a circle are unequal, the greater chord is nearer to

the center of the circle. Conversely, if two chords are unequally distant from the center, the nearer chord is the greater.

68. A line which is perpendicular to a radius of a circle at the outer end of that radius is tangent to the circle.
69. If a radius of a circle meets a tangent to the circle at the point of tangency, the radius is perpendicular to the tangent.
    A. If two circles are tangent to each other, their line of centers passes through the point of tangency.
70. Two tangents drawn to a circle from a given external point are equal, and they make equal angles with the line connecting the given point with the center of the circle.
71. If two chords intersect each other, the product of the segments of one equals the product of the segments of the other.
72. If a tangent and a secant are drawn to a circle from a given external point, the tangent is the mean proportional between the whole secant and the external segment of that secant.
    A. If from a given external point any secant is drawn to a given circle, the product of the whole secant and its external segment is constant.

XIII. *Angles and Circles*

73. A central angle is measured by its intercepted arc.
74. An inscribed angle is measured by one-half of its intercepted arc.
    A. An angle inscribed in a semicircle is a right angle.
75. The opposite angles of a cyclic quadrilateral are supplementary.
76. If the opposite angles of a quadrilateral are supplementary the quadrilateral is cyclic.
77. If two lines intersect each other within a circle, either angle thus formed is measured by one-half the sum of its two opposite intercepted arcs.
78. If two lines are drawn to a circle from a given external point, the angle thus formed is measured by one-half the difference of the arcs intercepted by this angle. (These lines may be two secants, two tangents, or a secant and a tangent.)
79. If a chord meets a tangent at the point of tangency, either angle between the chord and tangent is measured by one-half its intercepted arc.

XIV. *Loci* *

80. In a given plane the perpendicular bisector of a line-segment is the locus of points which are equidistant from the ends of the line-segment.

* Recall that if you are to prove that a certain line or curve $c$ is a locus under certain given conditions, then you must prove:

(1) Any point *on* $c$ satisfies the given conditions.

(2) Any point which satisfies the given conditions lies on $c$,— or, what is the same thing. Any point *not* on $c$ does not satisfy the given conditions.

A. Two points each equidistant from the ends of a given line-segment determine the perpendicular bisector of that line-segment, provided that the two given points and the given line-segment lie in one plane.

B. If two circles in the same plane intersect each other, their line of centers is the perpendicular bisector of their common chord.

81. In a given plane the bisector of an angle is the locus of points which are equidistant from the sides of the angle.

A. In a given plane the locus of points which are equidistant from two intersecting straight lines is a pair of lines which bisect the angles formed by the two intersecting lines.

82. If an angle with vertex $P$ is constant in size, and if its sides pass through two fixed points $A$ and $B$ — $P$ being and remaining on one side of the segment $AB$ — the locus of the point $P$ in a given plane is an arc of a circle which passes through the points $A$ and $B$.

XV. *Measurement Formulas.* (Triangles, polygons, circles)

83. The area of a rectangle is the product of its base and altitude.

84. The area of a parallelogram is the product of its base and altitude.

A. The area of a rhombus is one-half the product of its diagonals.

85. The area of a triangle is one-half the product of its base and altitude.

A. The area of a right triangle is one-half the product of its legs.

B. In a triangle $ABC$:

$$K = \tfrac{1}{2}(ab \sin C) = \tfrac{1}{2}(ac \sin B) = \tfrac{1}{2}(bc \sin A).$$

C. If two triangles have an angle in common, their areas have the same ratio as the products of the sides which include the common angle.

D. In a triangle $ABC$: $K = \sqrt{s(s-a)(s-b)(s-c)}$, where

$$s = \tfrac{1}{2}(a+b+c).$$

86. The area of a trapezoid is the product of the altitude and one-half the sum of the bases.

A. The area of a trapezoid is the product of its altitude and median.

87. The area of any regular polygon is one-half the product of its apothem and perimeter.

88. The circumference of a circle of radius $r$ is $2\pi r$.

89. The area of a circle is one-half the product of its radius and circumference, or $\pi r^2$.

90. In a circle of radius $r$, the area of a sector having a central angle of $\theta°$ is given by the formula:

$$K \quad \frac{\theta}{360}(\pi r^2).$$

A. In a circle of radius $r$, the area of a sector whose arc is $s$, $s$ being measured in linear units, is given by the formula:

$$K = \tfrac{1}{2}(rs).$$

XVI. *Limit Theorem from Higher Mathematics*

91. If two variable quantities are continually equal to each other as they approach their respective limits, their limits must be equal.

XVII. *Power Series from Higher Algebra*

92. The sum of the first $n$ terms of the power series

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$$

is given by the formula: $S = \frac{n}{6}(2n + 1)(n + 1)$.

A proof of this formula may be found by consulting any good textbook of Advanced Algebra. The most common method of proof is by means of Mathematical Induction.

The use of the formula is simple and direct. For example if we wish to find the sum of the first 12 terms of the series, substitute 12 for $n$ in the formula:

Thus: $S = \frac{12}{6}[2(12) + 1][(12) + 1] = 650.$

## AXIOMS

1. Quantities which are equal to the same quantity or to equal quantities are equal to each other.
2. A quantity may be substituted for its equal in an equation or in an inequality.
3. The whole equals the sum of all its parts, and is therefore greater than any of its parts.
4. Both members of an equation may be operated upon in the same manner mathematically without destroying the equality.
5. If unequals are added to unequals in the same order the results are unequal in the same order.
6. If unequals are subtracted from equals the results are unequal in reverse order.
7. If unequals are operated upon mathematically by positive equals in the same manner the results are unequal in the same order.
8. If $a > b$ and $b > c$, then $a > c$.

## POSTULATES

1. One and only one straight line can be drawn between two given points.
   A. Two given straight lines can intersect in not more than one point.
2. The shortest distance between two given points is a straight line.
3. All straight angles are equal.
   A. All right angles are equal.
   B. At a point on a given line one and only one line can be erected perpendicular to the given line — provided that both these lines are to lie in a specified plane.
   C. Equal angles have equal complements and equal supplements, and conversely.
   D. If two straight lines intersect each other the vertical angles are equal in pairs.
   E. If two angles are unequal the greater angle has the smaller complement and the smaller supplement, and conversely.
4. Through a given external point one and only one line can be drawn parallel to a given line.
   A. Lines which are parallel to the same line are parallel to each other — provided that all the lines lie in a specified plane.
5. If a regular polygon is inscribed in or circumscribed about a given circle, and if the number of sides of the polygon becomes infinite,
   i) the perimeter approaches the circumference of the circle as a limit;
   ii) the area approaches the area of the circle as a limit;
   iii) the apothem of the inscribed polygon, or the radius of the circumscribed polygon, approaches the radius of the circle as a limit.

# *Chapter One*

## PLANES

**1. Surface.** A precise definition of *surface* is difficult to state, and for present purposes is not necessary. In geometry we say that a point is that which has position only, having no length, width or thickness. We speak of a line as that which has length only, having no width or thickness. Correspondingly, we may regard a surface as that which possesses area but no thickness. In Plane Geometry we are accustomed to say that a line may be traced or generated by a moving point. Correspondingly, we say that a surface may be generated by a line or line-segment of some sort moving through space.

**2. Plane.** A plane is a surface such that a straight line connecting any two points of that surface lies completely within that surface.

In what respect does the surface illustrated in Fig. 4 fail to satisfy the definition of a plane?

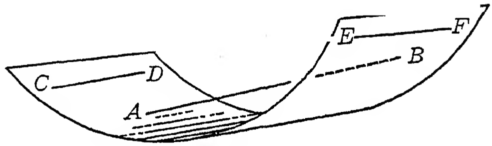



Fig. 4

**3. Representation of Planes.** A plane is of indefinite extent. However, when we draw a picture of a plane we usually indicate a finite portion of it, this portion being represented customarily by a rectangle or parallelogram. The usual way to designate a plane is by a single letter or possibly by two letters. Thus, in Fig. 5 we have a plane $N$ or a plane $RS$.

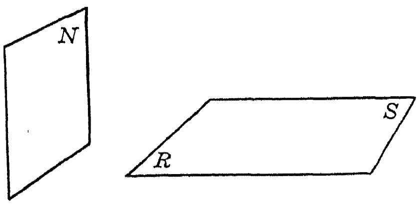



Fig. 5

**4.** Points are *collinear* if they lie in one straight line; otherwise they are non-collinear. Points or lines are *coplanar* if they lie in the same plane; otherwise they are non-coplanar.

**5.** Postulate 6. One and only one plane is determined by two given intersecting straight lines. (That is, one and only one plane can be drawn to contain two given intersecting straight lines.)

**6. Corollaries to Postulate 6.** One and only one plane is determined by

A. a given straight line and a point not on that line;

B. three given non-collinear points;
C. two given parallel lines.

Corollaries A and B follow at once from the postulate itself. Corollary C is obvious from the definition of parallel lines and from Postulate 4.

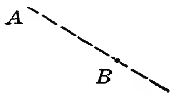



FIG. 6

7. POSTULATE 7. If two planes intersect each other, their intersection is a straight line.

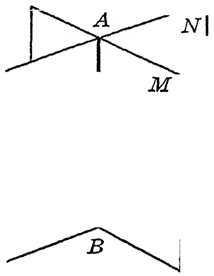



FIG. 7

## EXERCISES

*Group One*

1. If a piece of paper is folded why must the crease be straight?

2. Why will a three-legged stool always stand without rocking?

3. Which of the following are necessarily plane figures? a) parallelogram; b) triangle; c) circle; d) quadrilateral; e) rectangle; f) trapezoid; g) a circle with a line drawn tangent to it.

4. State a converse of Postulate 7. Is this converse true?

5. Under what conditions is the following statement true? "If two planes have three points in common, the two planes must coincide."

6. How many planes can be drawn to contain a given straight line $m$?

7. If a straight line $m$ intersects a plane $S$, how many points do $m$ and $S$ have in common?

8. $N$ and $S$ are two intersecting planes. A third plane $M$ intersects both $N$ and $S$. How many points are there which are common to $M$, $N$ and $S$? Discuss possible cases.

9. In general, if three lines $a$, $b$, $c$ are concurrent at a point $P$, how many planes are determined by these lines?

**10.** If two parallel lines $a$ and $b$ are each intersected by two other parallel lines $c$ and $d$, show that $a$, $b$, $c$, $d$ are coplanar.

**11.** Is it possible for a straight line-segment to move in space in such a way that it cannot generate a surface?

**12.** Tell which of the following statements is valid if the geometric figures mentioned are assumed to lie in space.

a) At a point in a line $m$ one and only one line can be drawn perpendicular to $m$.
b) From a given external point, one and only one line can be drawn perpendicular to a given line $m$.
c) Two lines perpendicular to the same line are parallel.
d) The sum of the angles of a triangle is two right angles.
e) If two straight lines intersect the vertical angles are equal.
f) Two triangles are similar if their corresponding sides are proportional.
g) Two triangles are congruent if $SAS = SAS$.
h) A line perpendicular to one of two parallel lines is perpendicular to the other, also.
i) The locus of points equidistant from the ends of a line-segment is the line which bisects this line-segment perpendicularly.
j) The diagonals of a parallelogram bisect each other.

**13.** Answer each of the following informally in your own words. No proof of any sort is expected.

a) $C$ is the mid-point of a fixed line-segment $AB$. $CD$ is a line which is perpendicular to $AB$. Let $CD$ rotate, always remaining perpendicular to $AB$. What sort of surface will $CD$ generate?

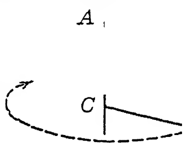



Fig. 8

b) $AB$ is a fixed line. $CD \parallel AB$, and is 2 in. from $AB$. Let $CD$ move around $AB$, always remaining parallel to $AB$ and 2 in. from it. What sort of surface will $CD$ generate?

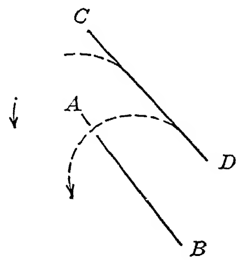



Fig. 9

c) Explain how a spherical surface (i.e., the surface of a ball) can be generated by some sort of moving line-segment.

d) How can the curved surface of a cone be generated by a moving line-segment?

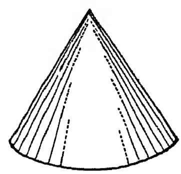


Fig. 10

e) The figure illustrated here has the shape of an inverted tub. Explain how the curved surface of the figure can be generated by a moving line-segment.

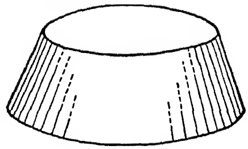


FIG. 11

**14.** How does a potter working at a potter's wheel illustrate the geometric concept of surfaces being generated by moving line-segments?

In the front of the book there is a reference list of theorems which have been established in Plane Geometry. In the text whenever allusion is made to one or more of these, the abbreviation "Ref. 63" or "Ref. 85–B," etc., will be used, meaning that you are to consult Theorem 63, Theorem 85–B, etc., in this list.

The following exercises require little or no technical knowledge of Solid Geometry. Their obvious purpose is to acquaint you with the task of visualizing and working with space diagrams. In three-dimensional drawings we have to employ "perspective" in our representations. It requires a certain amount of imagination to interpret a space drawing correctly. In turn, it often requires considerable care and ingenuity to create a space drawing which is clearly suggestive of the geometric figure which you are endeavoring to depict.

## EXERCISES

*Group Two*

**1.** At $A$, the mid-point of a line-segment $PQ$, any two line-segments $AB$ and $AC$ are drawn perpendicular to $PQ$. Draw $PB$, $PC$, $QB$, $QC$ and $BC$. Find three different pairs of congruent triangles. In each case prove the congruence.

$B$

FIG. 12

**2.** $AB$ lies in a plane $M$. Point $H$ is above $M$. $HD$ is perpendicular to $AB$ at $D$, the mid-point of $AB$. In plane $M$, line $DF$ is perpendicular to $AB$. $E$ is any point on $DF$. Draw $AE$ and $BE$. Prove that $\triangle HAE$ is congruent to $\triangle HBE$.

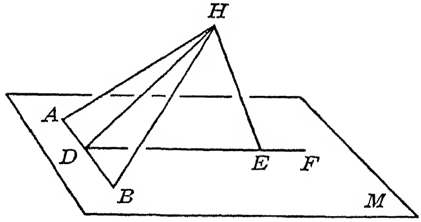



FIG. 13

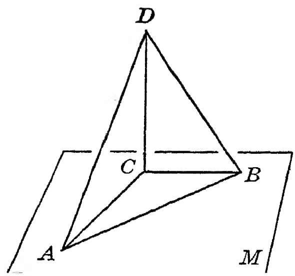



FIG. 14

3. $\triangle ABC$ lies in a plane $M$. $\angle ACB = 90°$. $CB = 6$ in., $CA = 8$ in. A line $DC$ is drawn so as to be perpendicular both to $CA$ and $CB$. $DC = 15$ in. Find the lengths of the sides of $\triangle ABD$.

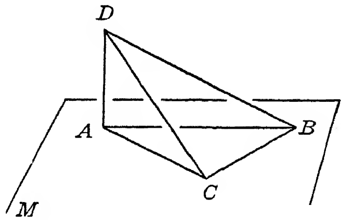



FIG. 15

4. $\triangle ABC$, right-angled at $C$, lies in a plane $M$. At $A$ a line $AD$ is drawn perpendicular to $AB$ and $AC$. Prove that $\triangle DCB$ is a right triangle by using Ref. 14.

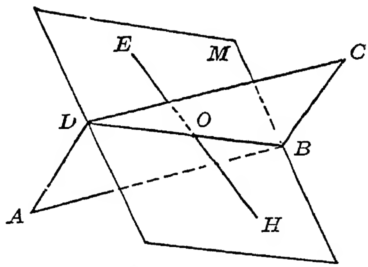



FIG. 16

5. A plane $M$ contains diagonal $DB$ of a parallelogram $ABCD$. In plane $M$ and through $O$, the intersection of the diagonals of $ABCD$, a line $EOH$ is drawn, bisected by point $O$. Draw $CH$ and $AE$. Prove $AE = CH$.

6. In Ex. 5 show that $CH$ and $AE$ are coplanar. Where is the intersection of this plane with the plane of $ABCD$?

7. In Ex. 5 draw $EC$ and $AH$ and show that $AHCE$ is a parallelogram.

8. $\triangle ABC$ lies in a plane $M$. $V$ is a point above the plane. $VA$, $VB$, $VC$ are drawn. $D$, $E$, $F$ are respectively the mid-points of $VA$, $VB$, $VC$. Prove:
a) $\triangle VDE \sim \triangle VAB$;
b) $\triangle VEF \sim \triangle VBC$;
c) $\triangle VFD \sim \triangle VCA$;
d) $\triangle DEF \sim \triangle ABC$.

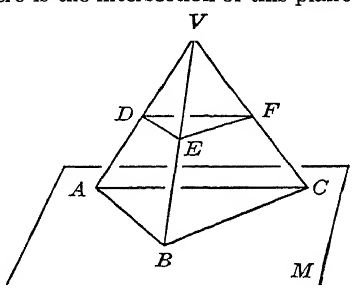



FIG. 17

**9.** In Ex. 8 if the area of $\triangle ABC$ is 20 sq. in., find the area of $\triangle DEF$.

**8. Two Common Errors.** In the preceding exercises you made use of certain facts established in Plane Geometry. In general that is the procedure throughout Solid Geometry. Be warned, however, against two very common and very natural logical errors.

(1) Before using a statement already established in Plane Geometry be certain that the truth of the statement does not break down when you work in space. For example, in space any number of lines can be drawn perpendicular to a given line at a given point on that line. Hence, the *unqualified statement:* "At a point in a line only one perpendicular can be erected to that line" is invalid in three-dimensional geometry.

(2) Before using a statement already established in Plane Geometry, even if you are certain that the statement is true when applied to space geometry, be sure that the *proof* of that statement when applied to a three-dimensional figure would be essentially no different from the proof used in Plane Geometry. If the proof of the three-dimensional case is necessarily different, then to employ this statement without further proof is a logical fallacy. For example, the *unqualified statement:* "Two angles with their sides parallel right-to-right, left-to-left are equal" has been proved only for the case when the two angles lie in the same plane. The statement is a *true* one in space, but when the angles are not in the same plane a different proof in general must be used.

# *Chapter Two*

## LINES PERPENDICULAR TO PLANES

**9. Line Perpendicular to Plane.** If a line $x$ meets a plane $M$ at a point $P$ in such a way as to be perpendicular to all lines in $M$ which pass through $P$, line $x$ is said to be *perpendicular* to the plane $M$.

Conversely, if a line is perpendicular to a plane, it must be perpendicular to all lines in that plane which pass through its foot. (The point of intersection of a line with a plane is often called the *foot* of the line.)

If a line is perpendicular to a plane, the plane is said to be perpendicular to the line.

If a line meets a plane and is not perpendicular to the plane, the line and plane are said to be *oblique* to each other.

**10.** THEOREM 1.

> If a line is perpendicular to each of two other lines at their point of intersection, it is perpendicular to the plane of those lines.

*Given:* $x \perp y$, $x \perp z$; $z$ meets $y$ at $P$; $z$ and $y$ determine a plane $M$. (See Fig. 18.)

*Prove:* $x \perp M$.

1) Let $w$ be any other line through $P$ in $M$. (See Fig. 19.) On $x$ take any point $A$. Extend $x$ through $M$ to a point $D$ so that $AP = PD$. In $M$ draw any line cutting $y$ at $B$, $z$ at $C$, $w$ at $E$. Draw $AB$, $AC$, $AE$, $DB$, $DC$, $DE$.
2) $CA = CD$ and $BA = BD$. Why?
3) Prove $\triangle ABC \cong \triangle DBC$ and obtain $\angle ABC = \angle DBC$.
4) Prove $\triangle ABE \cong \triangle DBE$ and obtain $EA = ED$.
5) Then $EP \perp AD$ (Ref. 80–A). That is, $x \perp w$.
6) But $w$ is *any* line in $M$ through $P$, other than $x$ and $y$.
7) Therefore, $x$, being perpendicular to any and all lines in $M$ which pass through $P$, must be perpendicular to $M$ (§ 9).

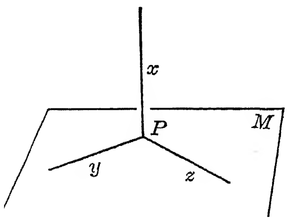



FIG. 18

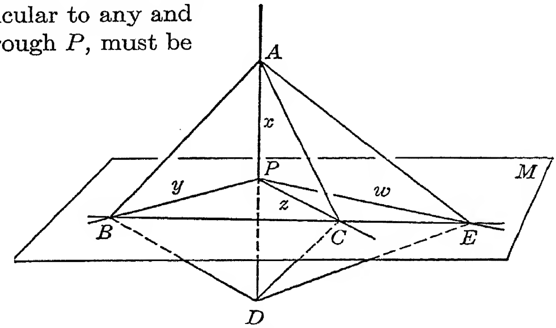



FIG. 19

**11. Constructions in Solid Geometry.** At this point it is desirable to mention the general problem of making constructions in Solid Geometry.

The treatment of constructions in space is necessarily more artificial than that of Plane Geometry constructions. For example, with pencil and paper we are unable to construct a plane or a spherical surface in the same sense that we actually construct a straight line or a circle by the use of straight-edge and compass. Several illustrations may help to clarify this idea.

*Example 1.* Given a line $x$ and an external point $P$. Construct a plane containing $x$ and $P$.

From § 6 we know that $x$ and $P$ determine one and only one plane.

From the great number of possible planes existing in space select that one which is determind by $x$ and $P$.

Indicate this plane (or a portion of it) in your diagram.

"Constructing" a plane, therefore, amounts to


FIG. 20A

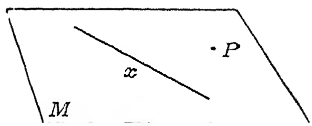



FIG. 20B

(1) having at hand or producing the proper combination of points or lines (§§ 5, 6) which will determine the plane;
(2) indicating this plane in the diagram.

*Example 2.* Given two points $A$ and $B$ in space. Construct line-segment $AB$.

Select any third point $C$. Construct the plane $M$ which is determined by $A$, $B$, $C$ (§ 6–B).

In plane $M$ draw $AB$.


FIG. 21A

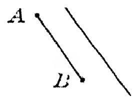



FIG. 21B

*Example 3.* Given a point $P$ on a line $x$ in space.

Construct a line $y$ which shall be perpendicular to $x$ at $P$.

There are innumerable planes in space which contain line $x$. Construct one of these planes. Call it the plane $M$.

Working now in plane $M$ we have the Plane Geometry problem of constructing a line which shall be perpendicular to $x$ at $P$.

It is not necessary to explain the details of making this construction, for it has already been demonstrated and proved in Plane Geometry. We are at liberty to use Plane Geometry material provided that we apply that material to one plane surface at a time. Therefore,

FIG. 22A

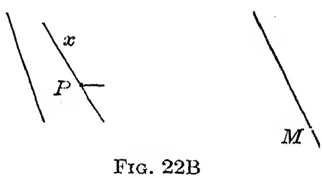



FIG. 22B

we merely say: "In plane $M$ construct $y$ perpendicular to $x$ at $P$." Finally, indicate the line $y$ in the diagram.

*Example 4.* Given an external point $P$ and a line $x$.

From $P$ construct a line $y$ which shall be perpendicular to $x$.

Study the diagram and work out the construction for yourself.

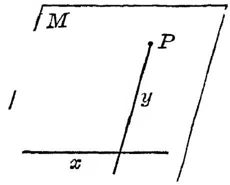


FIG. 23

The following fundamental constructions, each of which has been considered in the preceding illustrations, are listed for your convenience. Be able to perform these constructions. These constructions may be quoted as authorities whenever you are required to make other constructions later on.

**12.** CONSTRUCTION 1.

Construct a plane under any one of the conditions mentioned in §§ 5 and 6.

**13.** CONSTRUCTION 2.

Construct a straight line determined by two given points in space.

**14.** CONSTRUCTION 3.

At a point $P$ in a given line $x$ construct a line perpendicular to $x$.

**15.** CONSTRUCTION 4.

From an external point $P$ construct a line perpendicular to a given line $x$.

**16.** CONSTRUCTION 5.

Through a given external point $P$ construct a line $x$ which shall be parallel to a given line $y$.

Construct the plane $M$ determined by $P$ and $y$. Now work in plane $M$, recalling the corresponding construction in Plane Geometry.

**17.** CONSTRUCTION 6.

At a point $P$ in a given line $x$ construct a plane $M$ which shall be perpendicular to $x$.

At $P$ draw two lines $y$ and $z$ each perpendicular to $x$ (§ 14) Show that the plane $M$ determined by $y$ and $z$ is the required plane.

**18.** Construction 7.

Through a given external point $P$ construct a plane $M$ which shall be perpendicular to a given line $x$.

From $P$ draw line $y$ perpendicular to $x$ (§ 15). Let $y$ cut $x$ at a point $A$. At $A$ draw a line $z$ perpendicular to $x$ (§ 14). Show that the plane $M$ determined by $y$ and $z$ is the required plane.

**19.** Theorem 2.

Through a given point there is one and only one plane which is perpendicular to a given line.

From §§ 17, 18 there is at least one such plane. We are to show that there is *not more than one* such plane.

Let $P$ be the given point; let $x$ be the given line; let $M$ be one plane $\perp x$. There are two cases to be considered:

(a) when $P$ lies on $x$;

(b) when $P$ is an external point.

(a) $P$ lies on $x$ (Fig. 24).

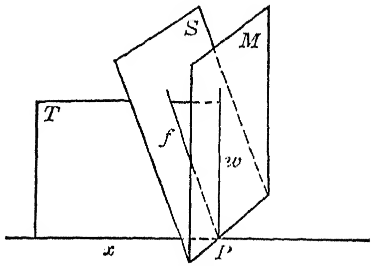


Fig. 24

1) Assume there is a second plane $S$ which is perpendicular to $x$ at $P$.
2) Draw any plane $T$ containing $x$ and cutting $M$ in line $w$ and $S$ in line $f$.
3) $\therefore$ $w$ and $f$ must both be perpendicular to $x$ (§ 9).
4) But since $w$ and $f$ are coplanar, they cannot both be perpendicular to $x$ at $P$ (Postulate 3-B).
5) $\therefore$ the assumption of a second plane $S$ perpendicular to $x$ at $P$ leads to a contradiction, and must therefore be false.
6) $\therefore$ $M$ is the only plane which is perpendicular to $x$ at $P$.

(b) $P$ is an external point (Fig. 25).

Assume that there is a second plane $S$ through $P$ and perpendicular to $x$. Apply the general method of proof used for case (a).

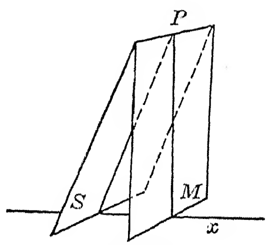


Fig. 25

**20. Corollary A** (Th. 2).

Any line which is perpendicular to a given line at a given point in that line must lie in the plane which is perpendicular to the given line at the given point.

*Given:* $y \perp x$ at $P$ in $x$; $M \perp x$ at $P$.

*Prove:* $y$ must lie in $M$.

Draw plane $S$ determined by $y$ and $x$; let $S$ cut $M$ in a line $y'$.

$y$ and $y'$ are each perpendicular to $x$ at the same point $P$; also, $y$ and $y'$ are coplanar.

Why must $y$ coincide with $y'$, and hence lie in plane $M$?

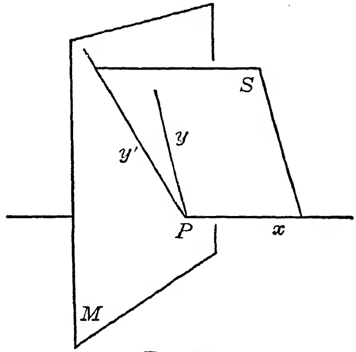


FIG. 26

**21.** CONSTRUCTION 8.

At a given point $P$ in a plane $M$ construct a line which shall be perpendicular to $M$.

In $M$ draw any line $x$ through $P$.

Through $P$ draw the plane $S$ which is perpendicular to $x$ (§ 17). Let $S$ cut $M$ in line $y$.

In $S$ draw line $z$ perpendicular to $y$ at point $P$.

Then $z$ is the required line.

*Proof:*

$S \perp x$, or $x \perp S$. $\therefore$ $x \perp z$.

That is, $z \perp x$.

But $z \perp y$, also.

$\therefore$ $z \perp M$ (§ 10).

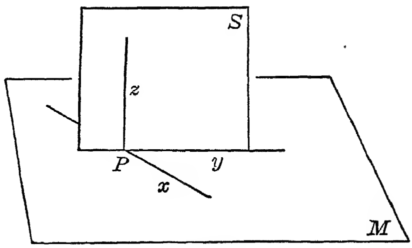


FIG. 27

**22.** CONSTRUCTION 9.

Through a given external point $P$ construct a line which shall be perpendicular to a given plane $M$.

In $M$ draw any line $x$. $P$ and $x$ determine a plane. From $P$ and in this plane draw $w$ perpendicular to $x$, meeting $x$ at a point $C$. In $M$ draw $t \perp x$ at $C$.

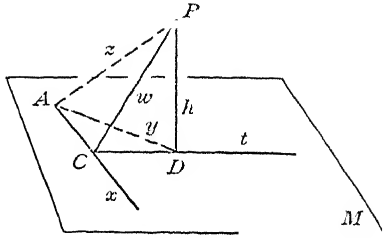


Fig. 28

From $P$ and in the plane of $w$ and $t$ draw $h$ perpendicular to $t$.

Line $h$ is the required line.

*Proof:*

On $x$ choose any point $A$ (other than $C$). Draw $y$ from $D$ to $A$, and $z$ from $P$ to $A$.

Triangles $PCA$, $ACD$, $CDP$ are right triangles by construction.

Show that $\triangle ADP$ is a right triangle by Ref. 14.

Then $h \perp y$. And since $h \perp t$, $\therefore h \perp M$ (§ 10).

**23.** Theorem 3.

> Through a given point there is one and only one line which is perpendicular to a given plane.

From §§ 21, 22 there is at least one such line. We are to show that there is *not more than one* such line.

Let $P$ be the given point; let $M$ be the given plane.

There are two cases: (a) $P$ lies in $M$; (b) $P$ is an external point.

(a) $P$ lies in $M$ (Fig. 29).

1) Let $z$ be $\perp M$ at $P$ (§ 21).
2) Assume there is a second line $t$ through $P$ which is also $\perp M$.
3) Draw plane $F$ determined by $z$ and $t$.
4) Obtain a contradiction to Postulate 3–B.

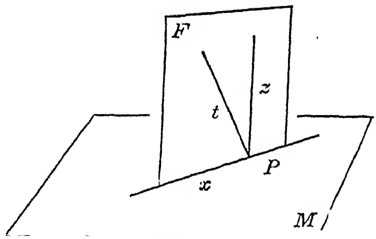


Fig. 29

(b) $P$ is an external point (Fig. 30).

Let $h$ be $\perp M$ (§ 22). Assume a second line $f$ from $P$ to be $\perp M$. Draw plane $S$ determined by $h$ and $f$, and obtain a contradiction to Ref. 32.

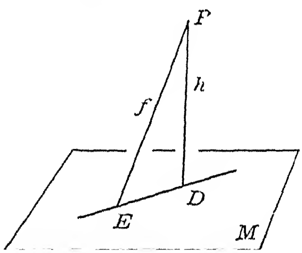


Fig. 30

**24. Corollary A** (Th. 3).

> The shortest distance from an external point to a given plane is the perpendicular from that point to the given plane.

**25. Distance from a Point to a Plane.** The distance from an external point to a plane is the perpendicular distance.

**26.** THEOREM 4.

> If from a point in a perpendicular to a plane two equal obliques are drawn to the plane, they meet the plane at points which are equidistant from the foot of the perpendicular, — and conversely. If from a point in a perpendicular to a plane two unequal obliques are drawn to a plane, the longer oblique meets the plane at a point farther away from the foot of the perpendicular than does the shorter oblique, — and conversely.

There are obviously four parts to the theorem. In each part we are given a line $x$ perpendicular to a plane $M$ at a point $D$. Point $A$ is the given point on $x$. The four parts are as follows:

| | *Given* | *Prove* |
|---|---|---|
| (a) | $y = w$ | $f = t$ |
| (b) | $f = t$ | $y = w$ |
| (c) | $y > w$ | $f > t$ |
| (d) | $f > t$ | $y > w$ |

The proof is easily accomplished by studying the right triangles.

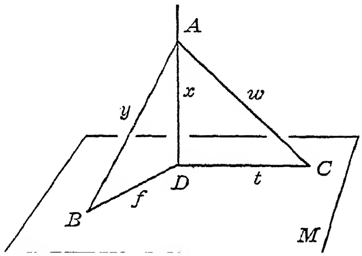


FIG. 31

**27.** THEOREM 5.

> If from the foot of a perpendicular to a plane a line is drawn at right angles to any line $w$ in that plane and meeting $w$ at $P$, then the line connecting $P$ with any point of the given perpendicular must be perpendicular to $w$.

*Given:* $x \perp M$ at $D$; $w$ any line in $M$ and not containing $D$; line $y$ through $D$ and $\perp w$ at $P$; $A$ any point on $x$.

*Prove:* $AP \perp w$.

Choose any point $B$ on $w$.

Draw $BD$ and $BA$, as shown.

Let $AB = h$, $DB = t$, $PB = z$, $AP = f$.

Show that $\angle APB$ is a right angle by showing that $h^2 = f^2 + z^2$.

*Note:* This theorem is sometimes referred to as "The Theorem of the Three Perpendiculars."

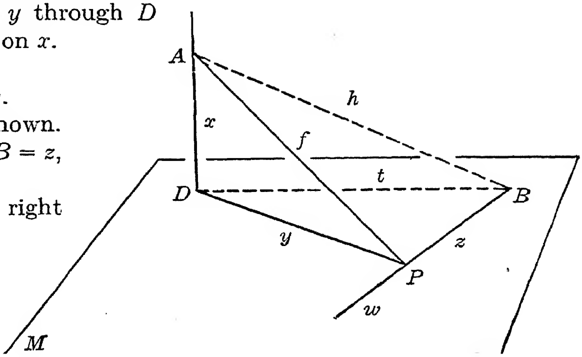


FIG. 32

## EXERCISES

*Group Three*

1. Prove Theorem 5 by the following method. On $w$ take $PE = PF$. Draw $DE$, $DF$, $AE$, $AF$. Use congruent triangles and Ref. 80–A.

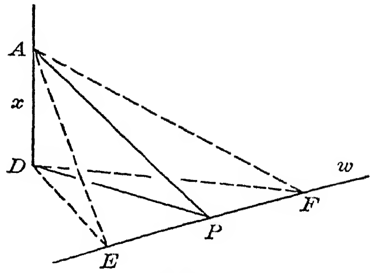



FIG. 33

2. A circle with center $O$ lies in a plane $M$. Line $h$ is perpendicular to $M$ at $O$. Line $t$ is tangent to circle $O$ at $A$. $B$ is any point on $h$. Prove: $BA \perp t$.

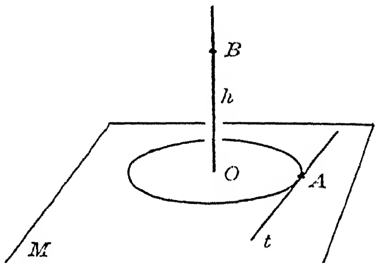



FIG. 34

3. A circle with center $O$ lies in a plane $M$. A line $h$ is perpendicular to $M$ at $O$. $AB$ is a chord of the circle. From $C$, any point on $h$, a line is drawn to bisect $AB$ at a point $E$. Prove: $CE \perp AB$.

4. In the figure of Ex. 2 show that $S$, the plane of $BO$ and $BA$, is perpendicular to $t$. In Ex. 3 show that $N$, the plane of $CO$ and $CE$, is perpendicular to chord $AB$

5. In this figure: $AC \perp M$ at $C$; $DE$ is a line in $M$; $AB \perp DE$ at $B$. Prove: $CB \perp DE$.

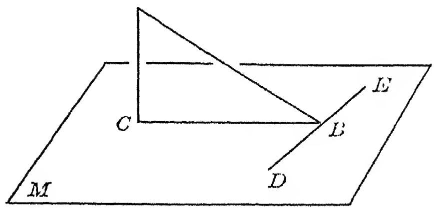



FIG. 35

6. In the accompanying figure $w$ and $z$ lie in a plane $M$. $w \perp z$ at $B$; $y \perp z$ at $B$. From any point $C$ on $y$, $x$ is dropped perpendicular to $w$, meeting $w$ at $A$. Prove: $x \perp M$.

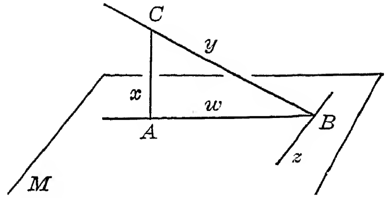



FIG. 36

7. $\triangle ABC$, right-angled at $C$, lies in a plane $M$. At $D$, the mid-point of $AB$, $DE$ is drawn perpendicular to $M$. Prove: $EA = EC = EB$.

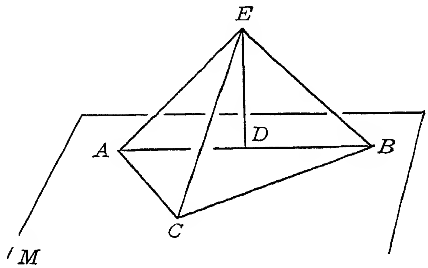



FIG. 37

8. In the preceding figure if $AC = 6$ in., $CB = 8$ in., $DE$ 12 in., find the lengths of $EA$, $EC$, $EB$, respectively. Find the area of $\triangle ACE$.

9. $\triangle ACD$, right-angled at $C$, lies in a plane $M$. $BD \perp M$ at $D$. $AC = 6$ in., $CD = 8$ in., $BD = 15$ in. Find the areas of triangles $BAD$ and $ACB$, and the length of $AB$.

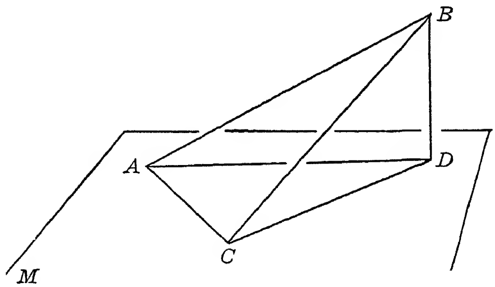



FIG. 38

10. $\triangle ABC$ lies in a plane $M$. At $O$, the circumcenter of $\triangle ABC$, line $h$ is perpendicular to $M$. Prove that any point $P$ on $h$ is equidistant from the points $A$, $B$, $C$

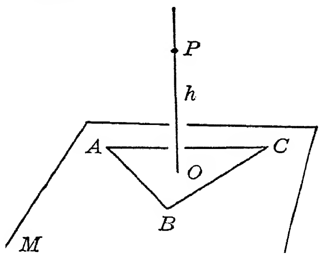



FIG. 39

**11.** In the figure for Ex. 10, let $D$ be any point which is equidistant from $A$, $B$, $C$, and is not necessarily in plane $M$. Prove that $D$ must lie on $h$. (From $D$ draw $x$ perpendicular to $M$, meeting $M$ at $E$. Show that $E$ coincides with $O$, and hence that $x$ coincides with $h$.)

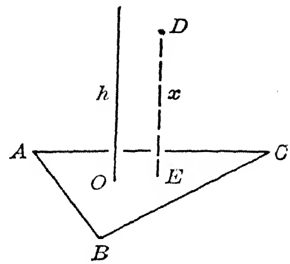


FIG. 40

**28.** THEOREM 6.

The locus of points which are equidistant from three given non-collinear points is the straight line which is perpendicular to the plane of the given points at the circumcenter of the triangle determined by these points.

(For proof see Exs. 10 and 11 above.)

*Note:* It is well to bear in mind constantly that in general if you are to show that some line or figure $c$ is a certain locus, then you must prove:
(a) any point on $c$ satisfies the conditions of the locus;
(b) any point which satisfies the conditions of the locus must lie on $c$, — or, what is the same thing, show that any point not on $c$ does not satisfy the conditions of the locus.

**29. Corollary A** (Th. 6).

Two points each equidistant from three given non-collinear points determine the straight line which is perpendicular to the plane of the given points at the circumcenter of the triangle determined by these points.

*Exercise:* What is the locus of points which are equidistant from all the points on the circumference of a given circle? Prove.

**30.** THEOREM 7.

The locus of points which are equidistant from two given points is the plane which bisects perpendicularly the line-segment joining the two points.

In the two accompanying figures, plane $M$ bisects $AB$ perpendicularly at the point $C$. You are to prove that $M$ contains all possible points which are equidistant from $A$ and $B$.

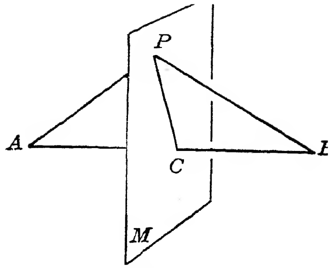



FIG. 41

(a)

Show that any point $P$ in $M$ is equidistant from $A$ and $B$.

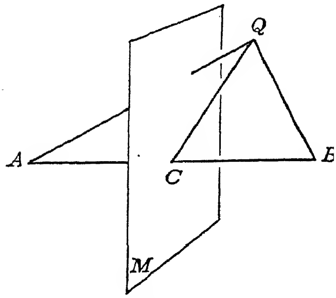



FIG. 42

(b)

Let $Q$ be any point known to be equidistant from $A$ and $B$. Draw $QA$, $QC$, $QB$. Show that $QC$ must lie in $M$, and hence that $Q$ itself must lie in $M$.

## EXERCISES

*Group Four*

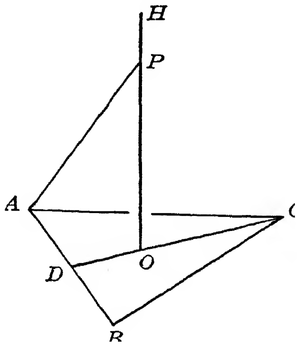



FIG. 43

1. Equilateral $\triangle ABC$ lies in a plane $M$. $HO \perp M$ at $O$, the centroid of $\triangle ABC$. Point $P$ is chosen so that $PA = AB$. Draw $PB$ and $PC$. Prove: $PA = PB = PC = AB$.

2. In the preceding figure show that the plane $S$ of points $P$, $D$, $C$ is perpendicular to $AB$.

3. In the preceding figure if $AB$  4 in., find the lengths of $DO$, $OC$, $PO$. Find the areas of triangles $PDC$ and $PAO$.

4. A $\triangle ABC$ lies in a plane $M$. Planes $R$, $S$, $T$ are respectively the perpendicular bisectors of the sides $AB$, $BC$, $CA$ of the triangle. Prove that the planes $R$, $S$, $T$ all meet in one common line; show that this common line is the perpendicular to $M$ erected at the circumcenter of $\triangle ABC$.

5. Point $D$ is above the plane $M$ of $\triangle ABC$. $P$ and $O$ are respectively the centroids of triangles $DAB$ and $ABC$. Show that lines $CP$ and $DO$ are coplanar and hence meet at some point $H$.

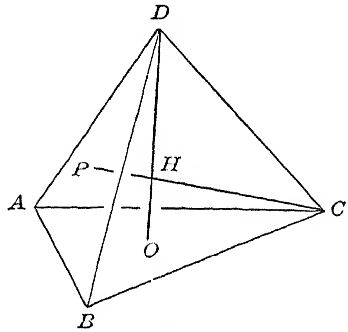



FIG. 44

6. In the preceding figure draw $PO$ and prove: $PO = \frac{1}{3}DC$, $PH = \frac{1}{4}PC$, $OH = \frac{1}{4}OD$. (Use similar triangles.)

7. In a figure like that of Ex. 5 assume that $AB = BC = CA = DA = DB = DC$. Assume that $P$ and $O$ are the centroids of triangles $DAB$ and $ABC$. Prove: $DO \perp M$, and $DO = CP$.

8. In Ex. 7 prove that plane $S$ of points $D$, $O$, $C$ bisects line $AB$ perpendicularly.

9. In Exs. 7 and 8 prove that $H$, the intersection of $CP$ and $DO$, is equidistant from the points $A$, $B$, $C$, $D$.

*Note:* The solid bounded by the equilateral triangles $ABC$, $DAB$, $DBC$, $DCA$ of the figure of Exs. 7–9 is called a *regular tetrahedron*. A more systematic study of the regular tetrahedron will come later in the book. When that point in the book is reached it will be advisable for you to return to this group of exercises.

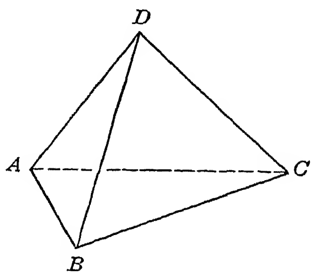



FIG. 45

**10.** Lines $x$, $y$, $w$ are any three lines lying in a plane $M$ and meeting one another at a common point $P$. A fourth line $h$ is drawn so that the angles which it makes with $x$, $y$, $w$ are equal. Prove: $h \perp M$.

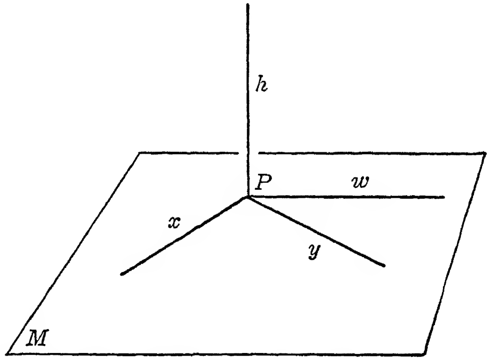



FIG. 46

# *Chapter Three*

## LINES PERPENDICULAR TO PLANES. LINES AND PLANES PARALLEL

**31. Parallel Lines.** Two straight lines are *parallel* if they lie in the same plane and do not meet however far they may be extended.

**32. Skew Lines.** Two straight lines are *skew* if they do not lie in the same plane and if they do not meet however far they may be extended.

**33. Line and Plane Parallel. Parallel Planes.** A line is parallel to a plane, or a plane is parallel to another plane, if the two do not meet however far they may be extended.

**34.** THEOREM 8.

Two lines perpendicular to the same plane are parallel.

*Given:* Lines $x$ and $y$ perpendicular to plane $M$ at $A$ and $B$, respectively.

*Prove:* $x \parallel y$.

From § 31 we must show:

(a) $x$ and $y$ are coplanar;

(b) $x$ and $y$ cannot meet.

After (a) is proved, (b) will follow if we can prove that $x$ and $y$ are perpendicular to the same line (Ref. 34).

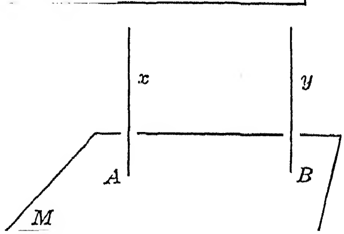


FIG. 47

(a)

1) In Fig. 47 draw $AB$. Through $B$ in $M$ draw $w$ perpendicular to $AB$. Let $C$ be any point on $x$. Draw $CB$ (Fig. 48).
2) $CB \perp w$.
3) $y \perp w$.
4) $AB \perp w$.
5) $\therefore AB, CB, y$ lie in some plane $S$. (See § 20.)
6) $\therefore x$ and $y$ are coplanar.

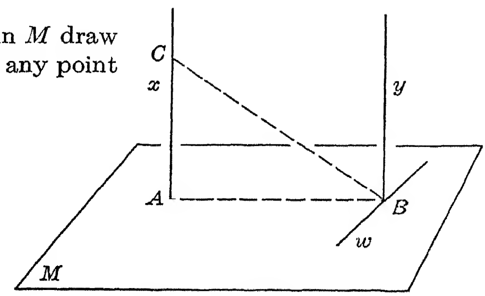


FIG. 48

(b)

Proof of (b) left to student.

**35. Corollary A** (Th. 8).

If a plane is perpendicular to one of two parallel lines, it is perpendicular to the other, also.

*Given:* $x \parallel y$; $x \perp M$.

*Prove:* $y \perp M$.

At $B$ draw $y'$ perpendicular to $M$. Show that $y'$ coincides with $y$. (See § 34 and Post. 4.)

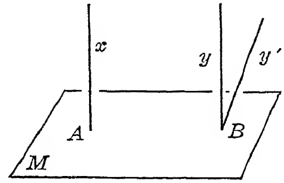



FIG. 49

**36. Corollary B** (Th. 8).

Two lines parallel to the same line are parallel to each other.

*Given:* $x \parallel h$; $y \parallel h$.

*Prove:* $x \parallel y$.

1) Through any point $A$ on $h$ draw a plane $M$ perpendicular to $h$.
2) $\therefore x \perp M$ and $y \perp M$ (§ 35).
3) $\therefore x \parallel y$ (§ 34).

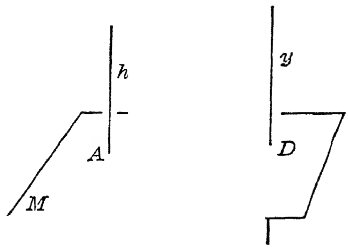



FIG. 50

37. THEOREM 9.

Two planes perpendicular to the same line are parallel.

*Given:* $M$ and $N$ perpendicular to $h$ at $A$ and $B$, respectively.

*Prove:* $M \parallel N$.

1) Suppose $M$ meets $N$, and that their intersection is a line $w$. Let $C$ be any point on $w$.
2) Then through $C$ there would be two planes $M$ and $N$ perpendicular to $h$.
3) But this contradicts § 19.
4) $\therefore$ the assumption of Step 1 is false, and hence $M$ cannot meet $N$.
5) $\therefore M \parallel N$ (§ 33).

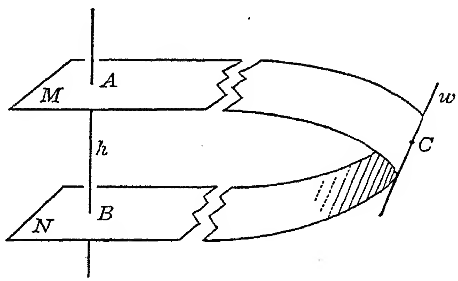



FIG. 51

**38.** THEOREM 10.

If two parallel planes are cut by a third plane, the two lines of intersection are parallel lines.

In the figure show that $x$ and $y$ are coplanar and that they can never meet.

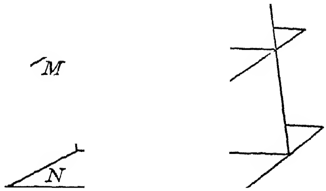



FIG. 52

**39.** THEOREM 11.

If a line is perpendicular to one of two parallel planes, it is perpendicular to the other, also.

*Given:* $M \parallel N$; $h \perp M$.

*Prove:* $h \perp N$.

Through $h$ draw any two planes, $R$ and $S$. $R$ cuts $M$ in $x$ and $N$ in $y$; $S$ cuts $M$ in $w$ and $N$ in $t$.

Show that $h \perp N$ by § 10.

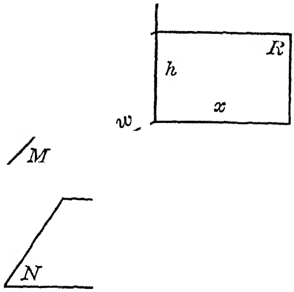



FIG. 53

**40. Corollary A** (Th. 11).

Two planes parallel to the same plane are parallel to each other.

Use §§ 39 and 37.

**41. Corollary B** (Th. 11).

The perpendicular distance between two given parallel planes is everywhere the same.

Use §§ 38 and 39.

**42. Distance between Two Parallel Planes.** The *distance* between two given parallel planes is the perpendicular distance, that is, the length of the common perpendicular (line) which is included between the planes.

**43.** THEOREM 12.

Three or more parallel planes have proportional intercepts on any two line transversals.

*Given:* $M \parallel N \parallel S$; $x$ and $y$ are any two transversals.

*Prove:* $\frac{AB}{BC} = \frac{DE}{EF}$.

1) Draw $AF$.
2) $x$ and $AF$ determine a plane $R$; $AF$ and $y$ determine a second plane $T$ in general, since $x$ and $y$ are not necessarily coplanar.
3) Now apply Ref. 40 to the line-segments in $R$, and again to those in $T$.

The segments on $AF$ will serve as a connecting link in obtaining the desired proportion between the segments on $x$ and those on $y$.

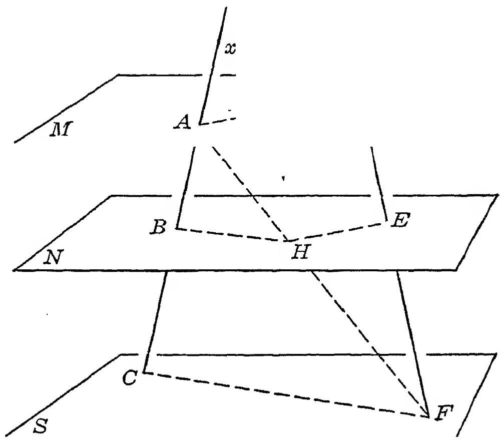



FIG. 54

**44.** THEOREM 13.

If an external line is parallel to a given line in a given plane, then the external line is parallel to the given plane.

*Given:* $y$ lies in $M$; $x \parallel y$.

*Prove:* $x \parallel M$.

1) If $x$ were to meet $M$ at some point $P$, then $P$ would have to be a point on line $y$, since $x$ and $y$ are coplanar.
2) But $x$ cannot meet $y$.
3) $x$ cannot meet $M$, and hence must be parallel to $M$.

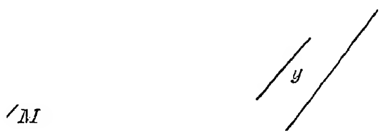



FIG. 55

**45.** CONSTRUCTION 10.

Through a given external point $P$ construct a line which shall be parallel to a given plane $M$.

In $M$ draw any line $w$.
In $S$, the plane of $P$ and $w$, draw a line $x$ through $P$ and parallel to $w$.
Line $x$ is the required line (§ 44).

**46.** Theorem 14.

If two intersecting lines are each parallel to a given plane, then the plane of those lines is parallel to the given plane.

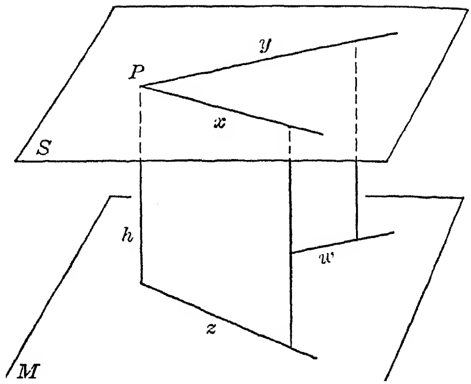



Fig. 56

*Given:* $x$ and $y$ intersecting at $P$. $S$ the plane of $x$ and $y$. $x \parallel M$; $y \parallel M$.

*Prove:* $S \parallel M$.

1) From $P$ draw $h \perp M$, cutting $M$ at $A$.
2) Draw $R$, the plane of $x$ and $h$, cutting $M$ in $z$; draw $T$, the plane of $y$ and $h$, cutting $M$ in $w$.
3) Prove $S \parallel M$ by § 37.

**47. Corollary A** (Th. 14).

Through a given external point there is one and only one plane which is parallel to a given plane.

From § 46 there is at least one such plane.

Assume that through the given point there is a second plane parallel to the given plane. From the given point draw a line perpendicular to the given plane. Obtain a contradiction to § 19.

**48. Corollary B** (Th. 14).

If a line is parallel to a plane, there is one and only one plane which contains this line and is parallel to the given plane.

(Proof left as an exercise.)

**49.** Construction 11.

Through a given external point $P$ construct a plane which shall be parallel to a given plane $M$.

Use either § 46 or § 37.

**50.** THEOREM 15.

If two angles not in the same plane have their sides parallel and extending in the same direction with reference to the line joining their vertices, the angles are equal and their planes are parallel.

*Given:* $\angle A$ in plane $S$; $\angle B$ in plane $M$; $x \parallel y$; $z \parallel w$.

*Prove:* (a) $\angle A = \angle B$;
(b) $S \parallel M$.

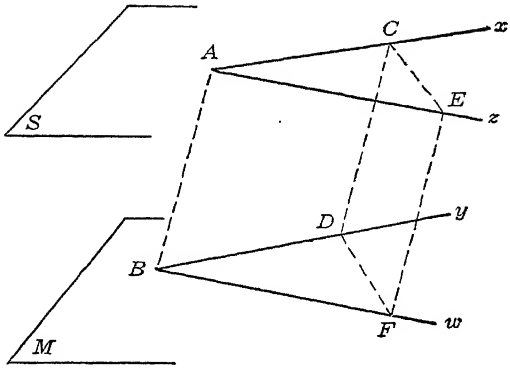



FIG. 57

(a)

1) Draw $AB$. On $x$ and $y$, respectively, take $AC = BD$. On $z$ and $w$, respectively, take $AE = BF$. Draw $CD$, $EF$, $DF$, $CE$.
2) Show that $ABFE$ and $ABDC$ are parallelograms. Thus obtain: $EF \parallel AB$, $CD \parallel AB$. Also obtain: $EF = AB$, $CD = AB$.
3) Now show that $CDFE$ is a parallelogram.
4) Prove $\triangle CAE \cong \triangle DBF$, and obtain $\angle A = \angle B$.

(b)

5) $M \parallel x$; $M \parallel z$ (§ 44). Or, what is the same thing, $x$ and $z$ are each parallel to $M$.
C) $\therefore$ $S$, the plane of $x$ and $z$, is parallel to $M$ (§ 46).

## EXERCISES

*Group Five*

**1.** In the following tell which statements are true and which are false:
(1) Two lines parallel to the same plane are parallel to each other.
(2) Two planes parallel to the same plane are parallel to each other.
(3) Two planes parallel to the same line are parallel to each other.
(4) If two planes are parallel, a line perpendicular to one plane is perpendicular to the other.
(5) If a line and a plane are parallel, a line perpendicular to the one is perpendicular to the other.
(6) If a line is parallel to a plane, it is parallel to every line in that plane.
(7) Two lines perpendicular to the same line are parallel.
(8) If a line $x$ and a plane $M$ are parallel to a line $y$, then $x$ is parallel to $M$.
(9) If a line is perpendicular to a plane, it makes equal angles with any two lines in that plane which pass through its foot.

(10) If a line intersects a plane so as to make equal angles with two lines in that plane which pass through its foot, the line is perpendicular to the plane.
(11) Through a given external point there is one and only one plane parallel to a given plane.
(12) Through a given external point there is one and only one line parallel to a given plane.
(13) If one of two intersecting lines is parallel to a given plane, the other line is also parallel to the given plane.
(14) If a line intersects one of two parallel planes, it intersects the other, also.
(15) If a line intersects one of two parallel lines, it intersects the other, also.
(16) If a plane intersects one of two parallel lines, it intersects the other, also.

**2.** If a given line is parallel to a given plane, and if a second plane containing this line intersects the given plane, then the given line is parallel to the intersection of the two planes. Prove.

**3.** If three lines all meet in one point and are cut by a fourth line, then all four lines are coplanar. Prove.

**4.** If a plane contains one of two parallel lines but not the other, show that the plane must be parallel to the other line.

**5.** Prove that if three or more parallel planes have equal intercepts on one line transversal, then they must have equal intercepts on any other line transversal.

**6.** Prove that if a line and a plane are each perpendicular to the same line, then the given line and the given plane are parallel.

**7.** Two lines $x$ and $y$ intersect each other at a point $P$ and determine a plane $M$. Through $P$ and *not* in $M$ show how to construct a line $w$ which shall make equal angles with $x$ and $y$.

**8.** Three non-coplanar lines $x$, $y$, $z$ are parallel to one another. $x$ and $y$ determine a plane $M$. Prove: $M \parallel z$.

**9.** A line $x$ is parallel to a plane $M$. Planes $S$ and $T$, each containing $x$, cut $M$ in $y$ and $z$, respectively. Prove: $y \parallel z$.

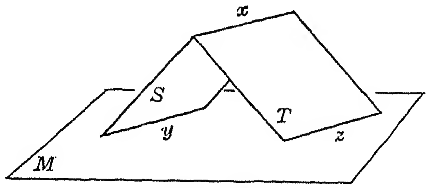



FIG. 58

**10.** Two planes $M$ and $N$ intersect in a line $x$. A plane $S$, parallel to $x$, cuts $M$ in a line $y$ and $N$ in a line $z$. Prove: $y \parallel x \parallel z$.

**11.** Two lines $x$ and $y$ are parallel. A plane containing $x$ but not $y$ intersects a plane $N$ containing $y$ but not $x$. $M$ intersects $N$ in a line $w$. Prove: $w \parallel x$ and $w \parallel y$.

**12.** Three parallel planes $M$, $N$, $S$ are cut by two lines $h$ and $t$. The intercepts on $h$ are 5 in. and 8 in., respectively; the corresponding intercepts on $t$ are 7 in. and $x$ in., respectively. Find $x$.

13. Four points $A$, $B$, $C$, $D$ are non-collinear and do not all lie in one plane. Points $E$, $F$, $G$, $H$ are respectively the mid-points of $AB$, $BC$, $CD$, $DA$. Prove that $E$, $F$, $G$, $H$ are coplanar (Ref. 28).

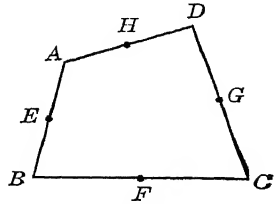



FIG. 59

14. In Ex. 13 show that $EG$ and $FH$ bisect each other.

15. A line $AB$ is perpendicular to a plane $M$ at $B$. $AB = 5$ in. In $M$ what is the locus of points which are 7 in. from $A$? Make a definite statement.

16. Complete the following statement: "The locus of points which are equidistant from two given parallel planes is . . ." Prove.

17. What is the locus of points which are equidistant from two given parallel lines? Prove your answer.

18. What is the locus of points which are 3 in. from a given plane?

19. $A$ and $B$ are two points above a plane $M$. What is the locus of points which are at the same time equidistant from $A$ and $B$ and $d$ in. from $M$? Answer without proof. Is the locus always possible? State any exceptional cases.

20. $x$ and $y$ are two parallel lines. $A$ and $B$ are two points not on $x$ or $y$ and not in the plane of $x$ and $y$. What is the locus of points which are at the same time equidistant from $x$ and $y$ and equidistant from $A$ and $B$? Answer without proof.

21. Points $A$ and $B$ are 21 in. apart. Describe definitely the locus of points which are at the same time 10 in. from $A$ and 17 in. from $B$. Answer without proof.

22. Point $D$ is not in the plane of $A$, $B$, $C$. Show how to locate a point which shall be equidistant from $A$, $B$, $C$, $D$. How many such points are there?

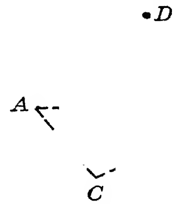



FIG. 60

23. $AO \perp M$ at $O$. $x$ is a line through $B$; $x$ and $B$ lie in $M$. From $A$ a line $y$ is drawn $\perp x$. $AO = 8$ in., $y = 10$ in. Given the plane $M$ and point $B$ and the line $AO$, show how to reproduce the lines $x$ and $y$.

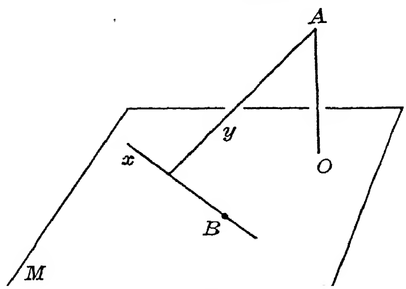



FIG. 61

**24.** $\triangle ABC$ lies in a plane $M$. Point $V$ is above $M$. A plane $S$, parallel to $M$, cuts $VA$, $VB$, $VC$ at points $D$, $E$, $F$, respectively. Prove: $\triangle VDE \sim \triangle VAB$, $\triangle DEF \sim \triangle ABC$.

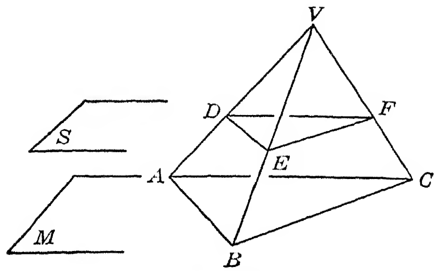



FIG. 62

**25.** In the preceding figure let $AB = 7$ in., $BC = 24$ in., $AC = 25$ in., $VD = 5$ in., $DA = 10$ in. Find the area of $\triangle DEF$.

**26.** In Ex. 25 find the area of the circle which can be circumscribed about $\triangle DEF$.

**27.** In Fig. 62 draw $VO$ perpendicular to $M$, cutting $M$ at $O$ and $S$ at $P$. Prove:

$$\frac{VO}{VP} = \frac{VC}{VF} = \frac{BC}{EF}.$$

**28.** In Fig. 62 let $H$ be the circumcenter of $\triangle ABC$. Draw $VH$, cutting $S$ at a point $J$. Prove that $J$ is the circumcenter of $\triangle DEF$.

**29.** In Fig. 62 let $K$ be the centroid of $\triangle ABC$. Draw $VK$, cutting $S$ at a point $Q$. Prove that $Q$ is the centroid of $\triangle DEF$.

**30.** In Fig. 62 draw a plane $T$ containing $VC$ and cutting $M$ in a line $w$ and $S$ in a line $f$. If $w$ bisects $\angle ACB$, prove that $f$ bisects $\angle DFE$.

**31.** In Ex. 30 let $AB = 12$ in., $BC = 10$ in., $AC = 8$ in., $VD = DA$. Find the lengths of the segments into which $f$ divides $DE$. (See Ref. 25.)

**32.** $A$ is a fixed point above two parallel planes $M$ and $N$. $O$ is a fixed point in $M$. $AO$ cuts $N$ at a point $B$. $Q$ is a moving point in $M$ and traces a circle about $O$ as a center. Draw $AQ$, cutting $N$ at $P$. Prove that in $N$ the point $P$ traces a circle about $B$ as a center.

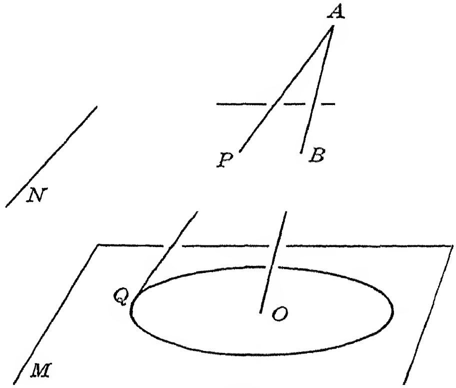



FIG. 63

**51.** CONSTRUCTION 12.

Construct a plane which shall contain one of two given skew lines and be parallel to the other.

$x$ is skew to $y$.
On $x$ choose any point $P$.
Through $P$ draw line $w$ parallel to $y$ (§ 16).
Now use § 44.


FIG. 64

**52.** THEOREM 16.

Through one of two skew lines there is one and only one plane which is parallel to the other line.

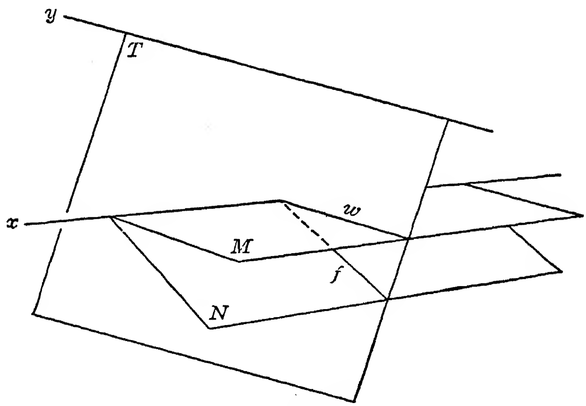



FIG. 65

*Given:* $x$ skew to $y$.
*Prove:* Through $x$ there is
(a) one plane parallel to $y$;
(b) not more than one plane parallel to $y$.

(a)

1) From § 51 there is at least one plane $M$ which contains $x$ and is parallel to $y$

(b)

2) Assume that there is a second plane $N$, containing $x$ and parallel to $y$.
3) Draw any plane $T$ containing $y$ and cutting $M$ in $w$ and $N$ in $f$.

4) $\therefore w \parallel y$ and $f \parallel y$. Why?
5) Show that the assumption of $N$ and $M$ *both* parallel to $y$ leads to a contradiction of Post. 4, and hence that $M$ is the *only* plane containing $x$ and parallel to $y$.

**53.** CONSTRUCTION 13.

| Through a given external point construct a plane which shall be parallel to two given skew lines. |
|---|

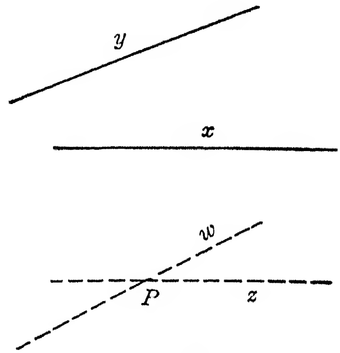



FIG. 66

$x$ is skew to $y$.

$P$ is an external point.

Through $P$ construct $z$ and $w$ parallel to $x$ and $y$, respectively.

Use § 44.

**54.** THEOREM 17.

| Through a given external point there is one and only one plane which is parallel to two given skew lines. |
|---|

The method of proof is similar to that used in § 52.

# *Chapter Four*

## DIHEDRAL ANGLES. PERPENDICULAR PLANES. PROJECTIONS

**55. Dihedral Angle.** A *dihedral angle* (dh $\angle$) is a figure formed by two planes meeting in a common line.

The planes themselves are the *faces*.
The common line is the *edge*.

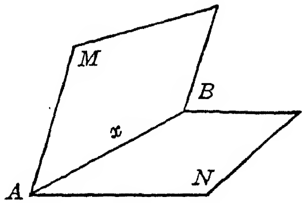



FIG. 67

**56.** Reference to a given dihedral angle is made by indicating the faces and the edge; or, when there is no ambiguity, a dihedral angle may be named by its edge only. In Fig. 67 the dihedral angle may be named as follows:

$$\text{dh}\angle M\text{–}AB\text{–}N;\quad \text{dh}\angle M\text{–}x\text{–}N;\quad \text{dh}\angle AB;\quad \text{dh}\angle x.$$

**57.** Fig. 68 represents a dh $\angle M$–$x$–$N$. In $M$ draw two lines $AB$ and $DE$ each perpendicular to the edge $x$. In $N$ draw $CB$ and $FE$ each perpendicular to $x$. By § 50, $\angle ABC = \angle DEF$ = any other angle similarly constructed. Hence, for a given dihedral angle all the angles such as $\angle ABC$ are constant in size.

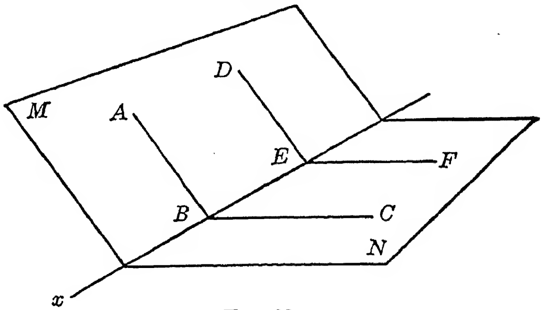



FIG. 68

**58. Plane Angle of a Dihedral Angle.** The *plane angle* of a dihedral angle is an angle whose vertex is on the edge of the dihedral angle and whose sides are perpendicular to the edge and lie one in each face of the dihedral angle. (See $\angle ABC$, for example, in Fig. 68.)

**59. Equal Dihedral Angles.** Two dihedral angles are *equal* if they can be made to coincide with each other. (If a plane containing the edge of a given dihedral angle divides the dihedral angle into two equal dihedral angles, this plane is said to *bisect* the given dihedral angle.)

**60.** ***Example:*** Prove that if two dihedral angles are equal, then their plane angles are equal. Conversely, show that if the plane angles of two dihedral angles are equal, then the dihedral angles are equal. (Both statements may be proved by the method of superposition.)

**61. Measure of a Dihedral Angle.** The *measurement* of a dihedral angle is the same as the measurement of its plane angle.

(Thus, "dh $\angle x = 32°$" means that the plane angle of dh $\angle x$ is 32°. Similarly, a *right* dihedral angle is one whose plane angle is a right angle; an *acute* dihedral angle is one whose plane angle is acute; an *obtuse* dihedral angle is one whose plane angle is obtuse.)

**62. Perpendicular Planes.** Two planes are *perpendicular* to each other if they meet so as to form a right dihedral angle.

**63.** Theorem 18.

If a line is perpendicular to a plane, then any plane containing that line is perpendicular to the given plane.

*Given:* $x \perp M$ at $A$; $S$ any plane containing $x$ and cutting $M$ in line $y$.

*Prove:* $S \perp M$.

1) In $M$ draw $w \perp y$ at $A$.
2) Show that dh $\angle S\text{–}y\text{–}M$ is a rt dh $\angle$ because its plane $\angle$ is a rt $\angle$.

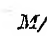



Fig. 69

**64.** Construction 14.

Through a given point which is either in a plane or outside the given plane construct a plane which shall be perpendicular to the given plane.

**65.** Theorem 19.

If two planes are perpendicular to each other, a line in one of these planes perpendicular to the line of intersection is perpendicular to the other plane.

*Given:* $M \perp N$, meeting $N$ in line $f$; $BC$ in $M$; $BC \perp f$ at $C$.

*Prove:* $BC \perp N$.

1) In $N$ draw $CD \perp f$.
2) How is $\angle BCD$ related to dh $\angle M\text{–}f\text{–}N$?
3) Show that $BC$ is perpendicular to $f$ and $CD$. Then why will $BC$ be perpendicular to $N$?

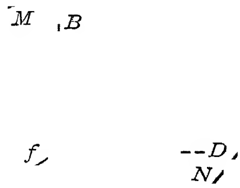



FIG. 70

**66. Corollary A** (Th. 19).

If two planes are perpendicular to each other, a line perpendicular to one of these planes at a point on their intersection must lie in the other plane.

*Given:* $M \perp N$, meeting $N$ in $w$; $A$ any point on $w$; $y \perp N$ at $A$.

*Prove:* $y$ lies in $M$.

At $A$ and in $M$ draw $x \perp w$. Show that $y$ must coincide with $x$ (§ 65).

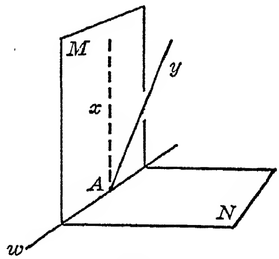



FIG. 71

**67. Corollary B** (Th. 19).

If two planes are perpendicular to each other, and if a line is drawn from any point in one of these planes perpendicular to the other plane, this line must lie in the first plane.

*Given:* $M \perp N$, meeting $N$ in $w$; $P$ any point in $M$; $PA \perp N$ at $A$.

*Prove:* $PA$ lies in $M$.

Use the method of § 66 above.

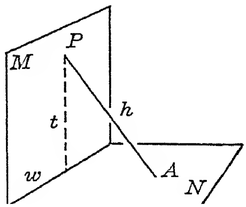



FIG. 72

**68.** THEOREM 20.

If two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to the third plane.

*Given:* $M$ and $N$ intersecting at $x$; $M \perp S$; $N \perp S$.

*Prove:* $x \perp S$.

1) From $P$ draw a line $h \perp S$.
2) $P$ must lie both in $M$ and $N$, since $x$ is the intersection of $M$ and $N$.
3) $\therefore$ $h$ must lie in $M$ (§ 67).
4) Similarly, $h$ lies in $N$.
5) Since $h$ lies in $M$ and $N$, $h$ must be the intersection of $M$ and $N$.
6) $h$ coincides with $x$, and hence $x$ must be perpendicular to $S$.

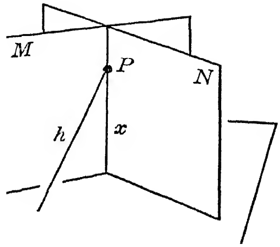



FIG. 73

**69.** THEOREM 21.

Through a given external line not perpendicular to a given plane there is one and only one plane perpendicular to the given plane.

*Given:* $x$ outside $M$; $x$ not perpendicular to $M$.

*Prove:* There is

(a) one plane $S$ containing $x$ and perpendicular to $M$;

(b) not more than one such plane.

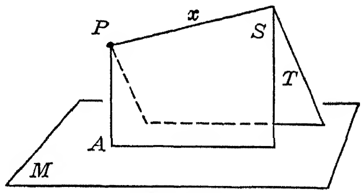



FIG. 74

(a)

From any point $P$ in $x$ draw $PA \perp M$. Now apply § 63.

(b)

Assume that there is a second plane $T$ containing $x$ and perpendicular to $M$. Then there would be two intersecting planes $S$ and $T$ each perpendicular to $M$. Therefore, by § 68, $x$ would have to be perpendicular to $M$. But $x$ is given *not* perpendicular to $M$. Hence, $T$ cannot be perpendicular to $M$.

**70. Projection of a Point.** If a point $P$ does not lie in a plane $M$, and if from $P$ a line is drawn perpendicular to $M$, meeting $M$ at a point $O$, then point $O$ is the *projection* of $P$ upon $M$.

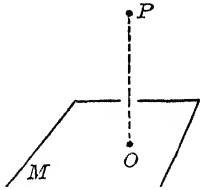



FIG. 75

**71. Projection of a Line.** The *projection* of any *line* or *curve* upon a given plane is the locus of the projections of the points of the given line upon the given plane.

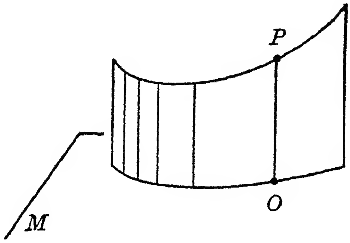



FIG. 76

72. THEOREM 22.

If a line $x$ not perpendicular to a given plane $M$ and not contained in $M$ is projected upon $M$, its projection upon $M$ is a straight line $y$. This line $y$ is the intersection of $M$ and the plane $S$ which contains $x$ and is perpendicular to $M$.

*Given:* Line $x$ outside a plane $M$; $S$ the plane containing $x$ and perpendicular to $M$; $S$ meets $M$ in a line $y$.

*Prove:* $y$ is the projection of $x$ upon $M$.

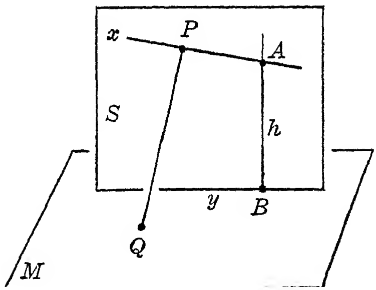



FIG. 77

(a)

Show that the projection of any point of $x$ upon $M$ must lie on the line $y$.

1) Let $Q$ be the projection of any point $P$ of line $x$ upon plane $M$.
2) Show that $PQ$ lies in $S$ (§ 67).
3) $\therefore$ $Q$ must lie on $y$.

(b)

Show that any point on $y$ is the projection of some point of $x$ upon $M$.

4) Let $B$ be any point on $y$.
5) At $B$ draw a line $h \perp M$.
6) $h$ must lie in $S$ (§ 66).
7) $\therefore$ $h$ cuts $x$ in some point $A$.
8) $\therefore$ $B$ is the projection of $A$ (a point of $x$) upon $M$.

From (a) and (b), line $y$ must be the locus of the projections of all points of $x$ upon $M$. Therefore, the line $y$ is the projection of the line $x$ upon the plane $M$ (§ 71).

**73.** THEOREM 23.

The acute angle which a straight line makes with its projection upon a given plane is smaller than the acute angle which the given line makes with any other line in the given plane.

*Given:* Line $x$ and plane $M$; $x$, extended if necessary, cuts $M$ at $A$; $p$ is the projection of $x$ upon $M$; $w$ is any other line through $A$ in $M$.

*Prove:* The acute angle between $x$ and $p$ is less than the acute angle between $x$ and $w$.

1) Choose any point $B$ on $x$.
2) From $B$ draw $BD \perp M$, meeting $M$ at $D$. Why does $D$ lie on $p$?
3) From $B$ draw $BC \perp w$, meeting $w$ at $C$.
4) $\therefore BD < BC$ (§ 24).
5) $\therefore \sin \angle BAD < \sin \angle BAC$.
6) $\therefore$, since $\measuredangle BAD$ and $BAC$ are each acute, $\angle BAD < \angle BAC$.

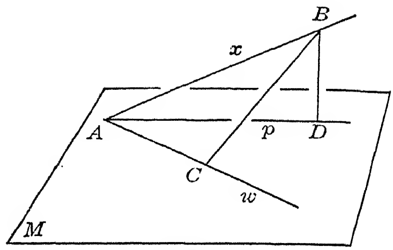



FIG. 78

**74. Inclination.** The acute angle which a given line makes with its projection upon a given plane is called the *angle* which the given line makes with the given plane. This angle is also called the *inclination* of the given line to the given plane.

**EXERCISES**

*Group Six*

1. Prove § 73 by the following method: On $x$ choose any point $B$. Draw $BD \perp M$, meeting $M$ at $D$. On $w$ take $AE = AD$. Apply Ref. 18 to triangles $BAD$ and $BAE$.

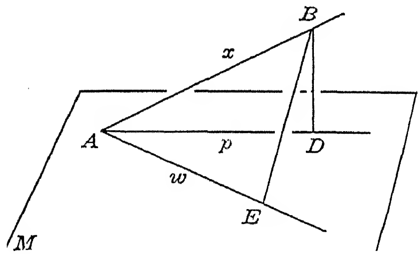



FIG. 79

2. $AB$ is any line-segment above a plane $M$. $AB$ is not perpendicular to $M$. From $A$ and $B$ draw $AC$ and $BD$, respectively, perpendicular to $M$. Draw $CD$. Show that $CD$ is the projection of $AB$ upon $M$.

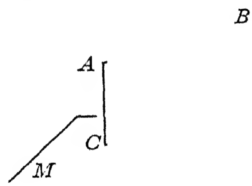



FIG. 80

3. A line and its projection upon a given plane determine a plane which is perpendicular to the given plane. Prove.

4. A line $x$ is outside a plane $M$. If $x$ and $M$ are each perpendicular to a second plane $S$, prove that $x$ is parallel to $M$.

5. If parallel lines intersect a plane they are equally inclined to the plane. Prove.

6. Parallel lines which are not perpendicular to a given plane and which are not in a plane perpendicular to the given plane have parallel projections upon the given plane. Prove.

7. If two parallel line-segments which are not perpendicular to a given plane and which do not lie in a plane perpendicular to the given plane are equal, then their projections upon the given plane are equal. Prove.

8. Three lines $x$, $y$, $z$ are concurrent at a point $A$. Each line is perpendicular to each of the other two. What can be said regarding each line and the plane of the other two?

9. In Ex. 8, if $M$ is the plane of $x$ and $y$, $N$ the plane of $x$ and $z$, $S$ the plane of $y$ and $z$, what can be said regarding the planes $M$, $N$, $S$?

10. A point $P$ and a line $x$ each lie outside a plane $M$. Show how to construct a plane which shall contain $P$, which shall be parallel to $x$, and which shall be perpendicular to $M$.

11. Prove that the legs of an isosceles triangle are equally inclined to a plane containing the base of the triangle.

12. If a line $x$, not perpendicular to a plane $M$, meets $M$ at $A$, prove that through $A$ and in $M$ there is one and only one line perpendicular to $x$.

13. If a line $x$ is parallel to a plane $M$, prove that any plane $S$ which is perpendicular to $x$ is also perpendicular to $M$.

14. If a plane $S$ is perpendicular to both faces of a dh $\angle M\text{-}x\text{-}N$, prove $S \perp x$.

15. A plane which is perpendicular to the edge of a given dihedral angle is perpendicular to both faces. Prove.

16. Show how to construct the plane which bisects a given dihedral angle. Prove that your construction is correct.

17. Three planes $M$, $N$, $S$ meet in a common line $x$. At any point $P$ on $x$ lines $h$, $f$, $t$ are drawn perpendicular to $M$, $N$, $S$, respectively. Prove that $h$, $f$, $t$ are coplanar.

18. $P$ is any point not in either face of a dh $\angle M\text{-}x\text{-}N$. Show how to construct a plane $S$ which shall be perpendicular to $M$ and $N$, and which shall contain $P$.

19. In Ex. 18 show how to construct through $P$ a line $w$ which shall be parallel to $M$ and $N$. Prove your construction.

20. Acute angle $ABC$ lies in a plane $M$. $BA = BC$. Planes $S$ and $T$ bisect perpendicularly the lines $BA$ and $BC$, respectively. Prove that $S$ and $T$ intersect each other in a line $h$ which is perpendicular to $M$ at a point on the bisector of $\angle ABC$ in $M$.

**21.** A parallelogram $ABCD$, not necessarily parallel to a plane $M$, lies above $M$ as shown. From the four vertices and from $E$, the intersection of the diagonals of $ABCD$, lines $x$, $y$, $z$, $w$, $h$ are drawn perpendicular to $M$. Prove: $h = \frac{1}{4}(x + y + z + w)$.

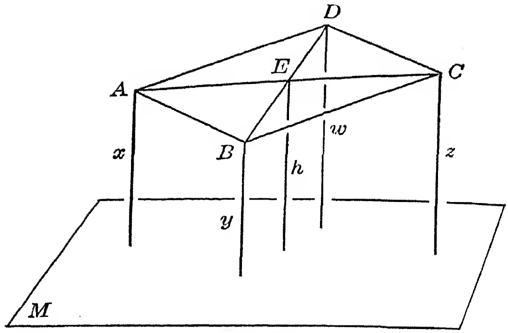



Fig. 81

**22.** Any plane $M$ is drawn to contain diagonal $DB$ of a parallelogram $ABCD$. $M$ is not perpendicular to the plane of $ABCD$. From $A$ and $C$, $AE$ and $CF$, respectively, are drawn perpendicular to $M$, meeting $M$ at $E$ and $F$. Prove: $AE = CF$.

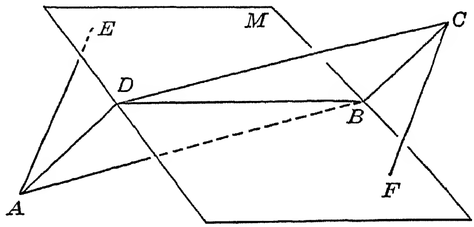



Fig. 82

**23.** Two planes $M$ and $N$ intersect each other in a line $EF$. In $M$, $BA \perp EF$; in $N$, $DC \perp EF$. $AB = DC$. $H$ and $J$ are the mid-points of $BD$ and $AC$, respectively. Prove that $HJ$ is perpendicular to $BD$ and $AC$.

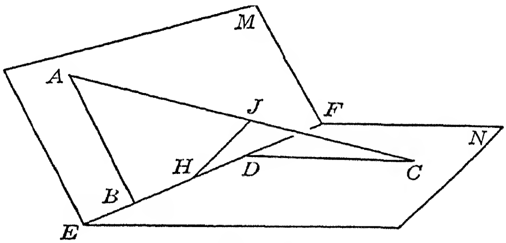



Fig. 83

**24.** $P$ is a point outside dh $\angle M\text{-}x\text{-}N$. $PA \perp M$, $PC \perp N$. $S$ is the plane of $PA$ and $PC$; $S$ cuts $x$ at $B$. Draw $PB$ and prove: $PB \perp x$.

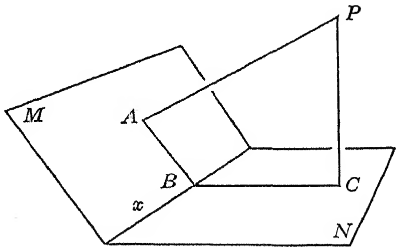



Fig. 84

**75.** Construction 15.

Construct the plane which bisects a given dihedral angle.

(See Ex. 16, Group Six.)

**76.** Theorem 24.

The locus of points equidistant from the faces of a given dihedral angle is the plane which bisects the given dihedral angle.

In Figs. 85 and 86, $S$ bisects dh $\angle M\text{–}x\text{–}N$.

(a) Let $P$ be any point in $S$.
Prove $P$ equidistant from $M$ and $N$.
(Draw $PA \perp M$, $PB \perp N$.)

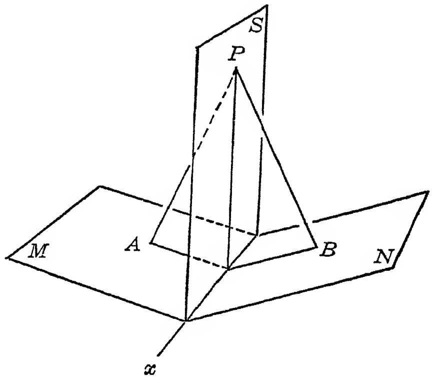



Fig. 85

(b) Let $Q$ be any point equidistant from $M$ and $N$.
Prove that $Q$ must lie in $S$.
(Draw $QC \perp M$, $QD \perp N$. Let plane of $QC$ and $QD$ cut $S$ in line $EQ'$. Show that $EQ$ must coincide with $EQ'$.)

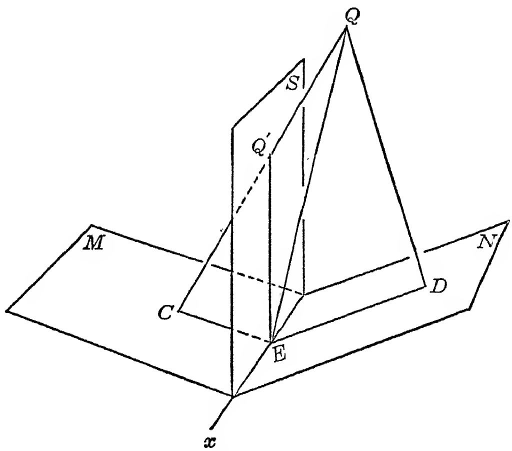



Fig. 86

**77.** CONSTRUCTION 16.

Construct a common perpendicular to two given skew lines.

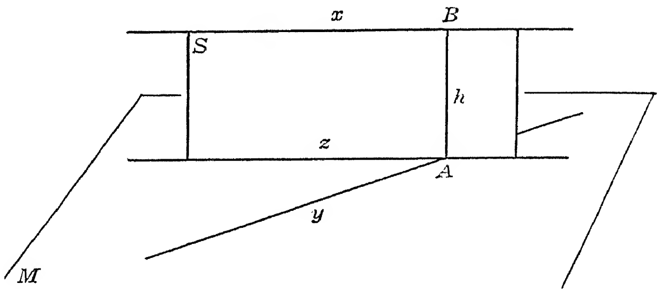



FIG. 87

$x$ is given skew to $y$.
1) Through $y$ construct the plane $M$ which is parallel to $x$ (§ 51).
2) Draw plane $S$ containing $x$ and perpendicular to $M$ (§ 64).
3) Let $S$ cut $M$ in $z$; let $S$ cut $y$ at $A$. Why is $x$ parallel to $z$?
4) In plane $S$ and at $A$ draw $h \perp z$.
5) Line $h$ is the required perpendicular.
In order to prove the construction see §§ 65, 9, and Ref. 35.

**78.** THEOREM 25.

If two lines are skew to each other there is one and only one line which is perpendicular to both skew lines; and this perpendicular is the shortest distance between the two given lines.

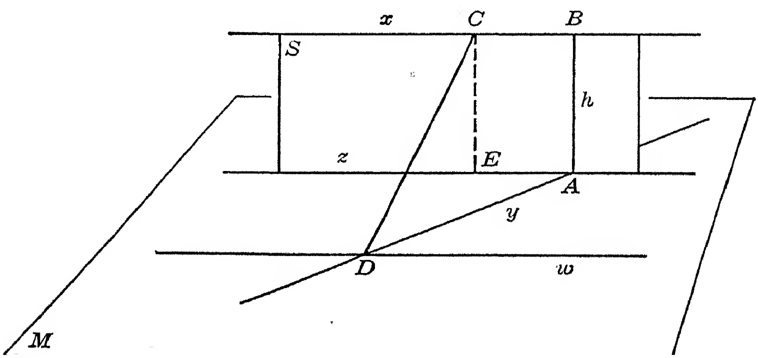



FIG. 88

*Given:* $x$ skew to $y$.

*Prove:* There is

(a) one common perpendicular to $x$ and $y$;
(b) not more than one such line.
(c) The common perpendicular is the shortest distance between $x$ and $y$.

(a)

1) From § 77 there is at least one line $h$ which is perpendicular to $x$ and $y$.

(b)

2) Assume that there is a second line $CD$ also perpendicular to $x$ and $y$.
3) As in § 77, let $M$ contain $y$ and be parallel to $x$; let $S$ contain $x$ and be perpendicular to $M$, cutting $M$ in line $z$.
4) In $M$ and through $D$ draw $w \parallel z$.
5) Since $CD \perp x$ by assumption, then $CD \perp w$, also (Ref. 35).
6) $CD \perp y$ by assumption.
7) $\therefore CD \perp M$.
8) In $S$ draw $CE \perp z$.
9) $\therefore CE \perp M$.
10) Thus from $C$ we appear to have two lines, $CD$ and $CE$, both perpendicular to $M$, — which contradicts § 23.
11) $\therefore$ the assumption made in step 2 must be false.
12) $\therefore h$ is the only common perpendicular to the skew lines $x$ and $y$.

(c)

Draw $BD$, for example, and show that $h$ is less than $BD$. In similar fashion $h$ can be shown to be less than any other line drawn from $x$ to $y$.

**79. Angle between Two Skew Lines.** Let $x$ and $y$ be two skew lines. Through $y$ draw the plane $M$ which is parallel to $x$. Let $p$ be the projection of $x$ upon $M$, cutting $y$ at some point $A$. The angle which $p$ makes with $y$ at point $A$ is often called the *angle between the skew lines* $x$ and $y$.

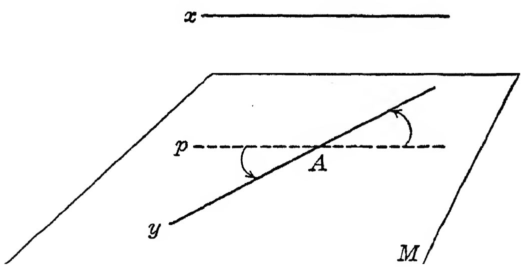



FIG. 89

Thus, if $p$ intersects $y$ at an angle of 30°, we say that the angle between the skew lines $x$ and $y$ is 30°. If the angle at $A$ is 90°, we may say that the skew lines $x$ and $y$ are "perpendicular" to each other.

## EXERCISES

*Group Seven*

**1.** $A$ and $B$ are any two points outside a dh $\angle M\text{-}x\text{-}N$. In general, what is the locus of points which are at the same time equidistant from $M$ and $N$ and equidistant from $A$ and $B$? Is the locus always possible?

**2.** Plane $S$ bisects dh $\angle M\text{-}x\text{-}N$. $\triangle ABC$ lies in $S$. Through points $A$, $B$, $C$ lines are drawn perpendicular to $S$. These lines cut $M$ at $D$, $E$, $F$, respectively; they cut $N$ at $X$, $Y$, $Z$, respectively. Prove: $\triangle DEF \cong \triangle XYZ$.

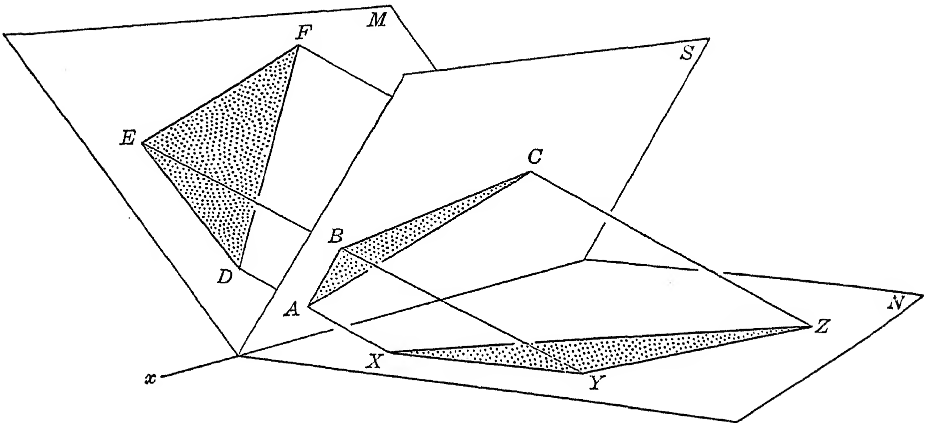



Fig. 90

**3.** Angle $ABC$ lies in a plane $M$. Plane $N$ is perpendicular to $M$, meeting $M$ in $BD$. $BD$ bisects $\angle ABC$. Prove that $N$ is the locus of points which are equidistant from $BA$ and $BC$.

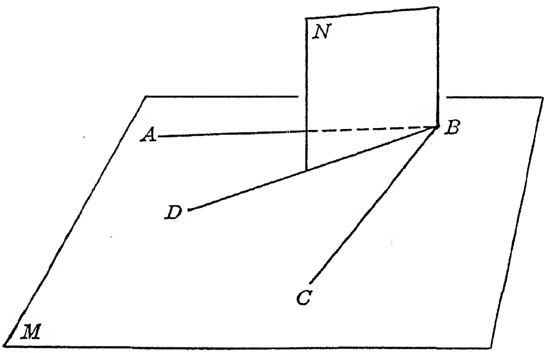



Fig. 91

4. Point $A$ lies in a plane $M$. $BC \perp M$ at $C$. $S \perp AB$ at $D$. $S$ cuts $M$ in $x$ and $AC$ at $E$. Prove: $x \perp AC$.

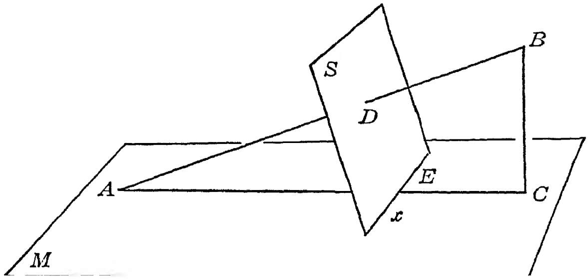



FIG. 92

5. A line $AB$ is perpendicular to a plane $M$ at $B$. A line $CD$ is perpendicular to $M$ at $D$. $AB = 32$ in., $BD = 32$ in., $CD = 8$ in. In $M$ show that the locus of points equidistant from $A$ and $C$ is a line $f$ which is perpendicular to $BD$ at some point $P$. Find the exact location of $P$ on $BD$.

6. $AB \perp M$ at $B$. $CD \perp M$ at $D$. Prove that the locus of points which are at the same time equidistant from $A$ and $B$ and equidistant from $AB$ and $CD$ is a line parallel to $M$, skew to $BD$ in general, and perpendicular to $BD$. (See § 79.)

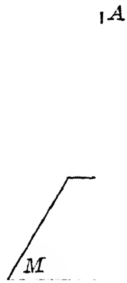



FIG. 93

7. Three lines $x$, $y$, $z$ are concurrent at a point $P$. Show how to construct a line $h$ which passes through $P$ and makes equal acute angles with $x$, $y$, $z$. (Assume that $x$, $y$, $z$ are not coplanar.)

8. Three planes $M$, $N$, $S$ intersect one another in the lines $x$, $y$, $z$. Prove that $x$, $y$, $z$ are parallel to one another, or else that $x$, $y$, $z$ must be concurrent.

9. $A$ and $B$ are any two points above a plane $M$. Show how to locate in $M$ a point $P$ such that the sum $(AP + PB)$ shall be as small as possible.

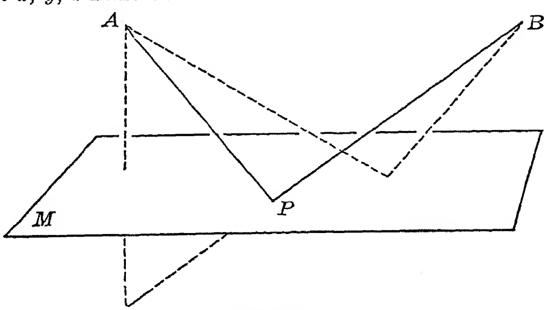



FIG. 94

**10.** Two planes $M$ and $N$ intersect each other in a line $x$. $\triangle ABC$ lies in $M$; $AB \parallel x$. $\triangle DEF$ is the projection of $\triangle ABC$ upon plane $N$. Let dh $\angle M\text{-}x\text{-}N = \theta$; let $k$ be the area of $\triangle ABC$; let $p$ be the area of $\triangle DEF$. Prove: $p = k \cos \theta$.

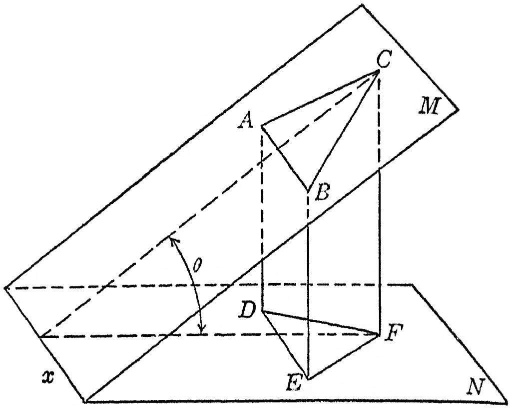



FIG. 95

**11.** Let $M\text{-}x\text{-}N$ be a dihedral angle as in Ex. 10. Let $k$ be the area of any polygon which lies in $M$; let $p$ be the area of the projection of the polygon upon plane $N$. Use the result of Ex. 10 to prove: $p = k \cos \theta$. (Show that the polygon in $M$ can be separated into a number of triangles with one side of each parallel to line $x$.)

**80.** Let $k$ be the area of any figure whatsoever which may lie in plane $M$. $M$ intersects $N$ in $x$, making dh $\angle M\text{-}x\text{-}N = \theta$. Let $p$ be the area of the projection of the given figure upon $N$. It can be proved that

$$p = k \cos \theta.$$

(The proof of this general case will not be discussed in this book. The truth of § 80 will be assumed.)

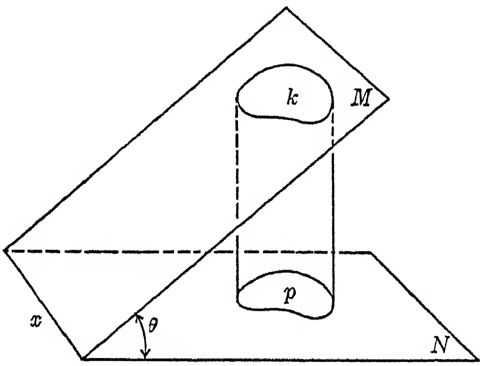



FIG. 96

# *Chapter Five*

## PRISMS

**81. Solid.** A solid is often called a finite portion of space, meaning that it is a portion of space having definite boundaries. These boundaries are surfaces, or perhaps a single surface, as in the case of a sphere. When a solid is bounded by several surfaces these surfaces are called the *faces* of the solid; the lines of intersection of the faces are the *edges;* the points of intersection of the edges are the *vertices*. The *area* of a solid is the sum of the areas of its faces. The *volume* is the amount of space enclosed by the bounding surfaces.

Just as a surface may be generated by a line moving through space (§ 1), so a solid may be generated by the motion of a surface through space.

**82. Plane Section of a Solid.** If a plane $M$ intersects a solid, the figure determined upon $M$ by the solid is called a plane section of the given solid.

**83. Prismatic Surface.** Let $p$ be any plane polygon. Let $x$ be a straight line touching $p$ but not coplanar with $p$. Let $x$ move so that it is always in contact with $p$ and so that it remains parallel to its original position. The surface traced by $x$ is called a *prismatic surface* (Figs. 97, 98).

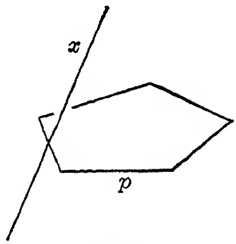



FIG. 97

Line $x$ is the *generatrix* of the surface; $p$ is the *directrix*. An *element* of the primastic surface is the line $x$ in any one of its innumerable positions. Clearly, a prismatic surface consists of three or more plane surfaces which may be called the *faces*. The intersections of these faces are the *edges*.

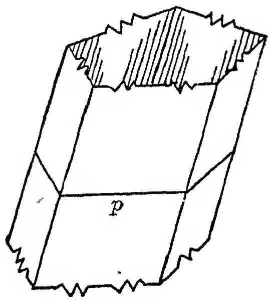



FIG. 98

**84. Prism.** If two parallel planes $M$ and $N$ cut all the edges of a prismatic surface, the solid bounded by $M$, $N$ and the prismatic surface is called a *prism*.

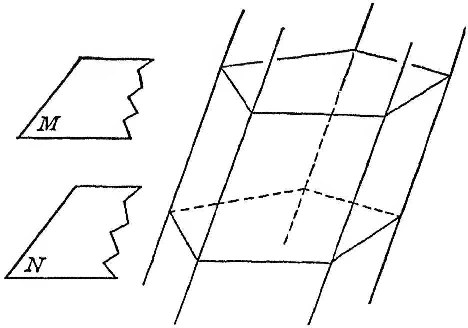



FIG. 99

**85.** Figure 100 shows a prism $ABCDE\text{–}A'B'C'D'E'$.

The following facts regarding this prism are easily proved.

A. The plane figures $ABB'A'$, $BCC'B'$, etc., are parallelograms.

B. The line-segments $A'A$, $B'B$, $C'C$, etc., are all equal.

C. The figures $ABCDE$ and $A'B'C'D'E'$ are congruent.

A prism may, therefore, be described as a solid whose bounding surfaces are:

(a) two congruent polygons in parallel planes with corresponding edges parallel; and

(b) three or more parallelograms.

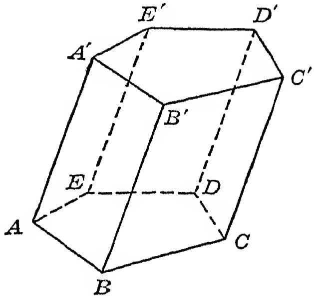



FIG. 100

**86. Parts of a Prism.** (Cf. Fig. 100.)

The *lateral edges* are the line-segments $AA'$, $BB'$, $CC'$, etc.

The *lateral faces* are the parallelograms $ABB'A'$, $BCC'B'$, $CDD'C'$, etc.

The *bases* are the polygons $ABCDE$ and $A'B'C'D'E'$.

The *basal edges* are $AB$, $BC$, $CD$, . . . $A'B'$, $B'C'$, $C'D'$, etc.

*Corresponding vertices* are $A$ and $A'$, $B$ and $B'$, $C$ and $C'$, etc.

*Corresponding basal edges* are $AB$ and $A'B'$, $BC$ and $B'C'$, $CD$ and $C'D'$.

The *altitude* of a prism is the perpendicular distance between the bases.

**87. Truncated Prism.** If a plane not parallel to the bases of a prism and not intersecting the bases cuts all the lateral edges, it divides the given prism into two solids, either of which is called a *truncated prism* ($ABC\text{–}RST$ or $RST\text{–}DEF$ in Fig. 101).

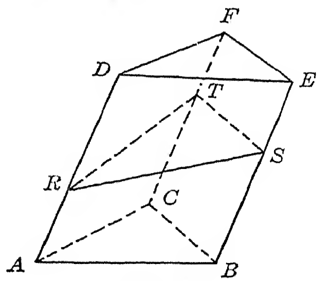



FIG. 101

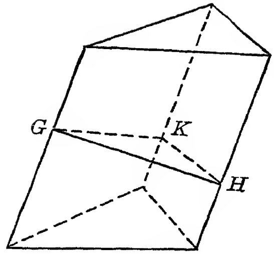


FIG. 102

**88. Right Section of a Prism.** If a plane cuts all the lateral edges of a prism perpendicularly, the plane section thus formed is called a *right section* of the prism ($\triangle GHK$ in Fig. 102).

**89. General Classification of Prisms.**

$$\text{A prism is} \left\{\begin{array}{l}\text{triangular}\\ \text{quadrangular}\\ \text{pentagonal}\\ \text{hexagonal}\end{array}\right\} \text{if its bases are} \left\{\begin{array}{l}\text{triangles}\\ \text{quadrilaterals}\\ \text{pentagons}\\ \text{hexagons}\end{array}\right\}\text{, etc.}$$

A *right prism* is one whose lateral edges are perpendicular to the planes of the bases.

An *oblique prism* is one which is not a right prism.

A *regular prism* is a right prism whose bases are regular polygons.

**90. Parallelepiped.** A parallelepiped is a prism whose bases are parallelograms.

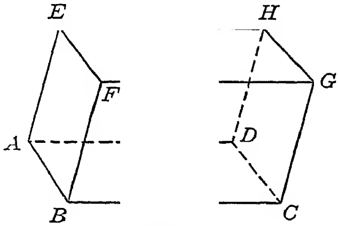



FIG. 103

Figure 103 shows a parallelepiped $ABCD$–$EFGH$. Lines $AG$, $EC$, $HB$, $DF$ are the *diagonals*. Planes $ACGE$, $HEBC$, etc., are the *diagonal planes*.

$$\text{A} \left\{\begin{array}{l}\text{rectangular parallelepiped}\\ \text{rectangular solid}\end{array}\right\} \text{is a right parallelepiped with rectangular bases.}$$

$$\left\{\begin{array}{l}\text{regular parallelepiped}\\ \text{regular rectangular solid}\\ \text{cube}\end{array}\right\} \text{is a rectangular solid all of whose edges are equal.}$$

**91.** THEOREM 26.

Sections of a prism made by two parallel planes cutting all the lateral edges of the prism are congruent; and these parallel sections themselves determine a prism.

(Proof left to student.)

**92.** THEOREM 27.

The opposite faces of a parallelepiped are congruent and their planes are parallel.

(For example show that $ABFE \cong DCGH$, and that their planes are parallel.)

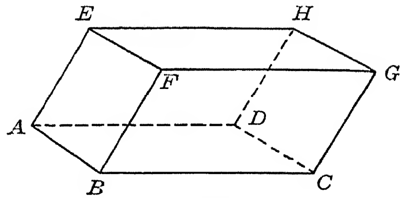



FIG. 104

**93.** THEOREM 28.

The diagonals of any parallelepiped are concurrent and bisect one another.

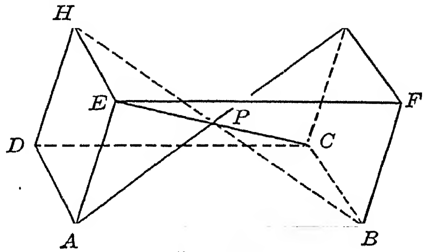



FIG. 105

1) Draw two diagonals $EC$, $AG$.
2) $ACGE$ is a parallelogram.
3) $\therefore$ $EC$, $AG$ are coplanar and bisect each other at a point $P$. In other words, $AG$ cuts $EC$ at $P$, the mid-point of $EC$.
4) Draw a third diagonal $HB$.
5) Show that $HB$ and $EC$ are coplanar (plane $HEBC$), and hence intersect and also bisect each other.
6) Thus far we have: $AG$ and $HB$ each passing through a common point $P$ on $EC$.
7) Complete the proof by considering diagonal $DF$. We shall then have all four diagonals passing through the same point $P$.

*Note:* The point $P$ above is called the *center* of the parallelepiped.

### EXERCISES

*Group Eight*

*Note:* If it is desirable, Exercises 1–8 may be treated as theorems.

**1.** Sections of a prism which are parallel to the bases are congruent to the bases.

**2.** All right sections of a given prism are congruent.

**3.** In a right prism the altitude equals any lateral edge.

**4.** The lateral faces of a regular prism are congruent rectangles.

5. In any prism a plane section determined by any two non-adjacent lateral edges is a parallelogram.

6. A section of a parallelepiped determined by any diagonal plane is a parallelogram.

7. The diagonals of a rectangular solid are equal.

8. In a rectangular solid a diagonal plane is perpendicular to two opposite faces.

9. If any two opposite faces of a parallelepiped are designated as bases, will the solid still conform to the definition of a parallelepiped in every respect?

10. If one edge of a cube is $x$, show that the length of a diagonal is $x\sqrt{3}$.

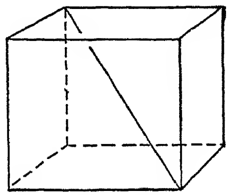


FIG. 106

11. A diagonal of a cube is 24 in. Find the length of one edge.

12. The sum of the areas of the faces of a cube is 36 sq. in. Find the length of a diagonal of one face of the cube. Find the length of one diagonal of the cube itself.

13. Find to the nearest tenth of a degree, or to the nearest minute, the number of degrees in the acute angle at which any two diagonals of a cube intersect each other. (This is best done by using the Law of Cosines from Plane Trigonometry.)

14. $ABCD$–$EFGH$ is a rectangular solid. $AB = 12$ in., $BF = 4$ in., $BC = 3$ in. Find the length of a diagonal of the solid.

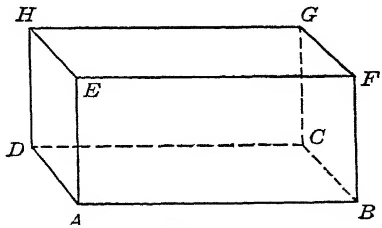



FIG. 107

15. In Ex. 14, find the area of section $ACGE$.

16. If $d$ is a diagonal of a rectangular solid and $a$, $b$, $c$ are three non-parallel edges, prove:

$$d = \sqrt{a^2 + b^2 + c^2}.$$

17. Three non-parallel edges of a rectangular solid are proportional to the numbers 3, 4, 5. One diagonal of the solid is $15\sqrt{2}$ in. Find the length of each edge.

18. In any prism, if a plane containing a lateral edge intersects a lateral face, the plane section thus formed is a parallelogram. Prove.

**19.** $ABC$–$DEF$ is a triangular prism. A plane containing edge $AB$ intersects base $DEF$ in line $HK$. Prove that section $ABHK$ is a trapezoid.

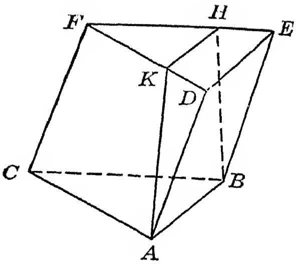



FIG. 108

**20.** $ABC$–$DEF$ is a right prism whose bases are equilateral triangles. A plane parallel to a basal edge but not parallel to the planes of the bases cuts all the lateral edges. Prove that the section thus formed is an isosceles triangle.

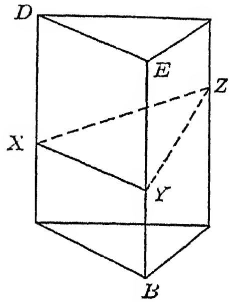



FIG. 109

**21.** $ABC$–$DEF$ is any triangular prism. $P$ and $Q$ are respectively the mid-points of $AB$ and $BC$. Prove that $DQ$ and $FP$ intersect each other at some point $O$. Also, show that

$$DO = \tfrac{2}{3}DQ \quad \text{and} \quad FO = \tfrac{2}{3}FP.$$

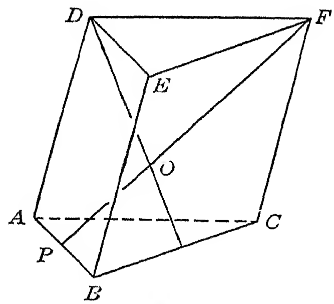



FIG. 110

**22.** In Ex. 21, let $H$ be the mid-point of $AC$. Draw $EH$, and prove that $EH$ passes through the point $O$. What is the ratio of $EO$ to $EH$?

**23.** In Ex. 21, draw through $O$ a line parallel to the lateral edges, cutting the top base at $K$ and the lower base at $J$. Prove that $K$ and $J$ are the centroids of triangles $DEF$ and $ABC$, respectively.

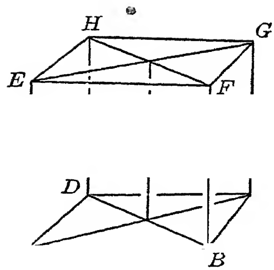


FIG. 111

24. In the cube $ABCD–EFGH$ prove that the diagonal planes $ACGE$ and $BFHD$ are perpendicular to each other.

25. In the preceding figure draw $AF$, $AH$, $HF$, $EC$. Prove that $EC$ is perpendicular to the plane of $\triangle AFH$ at the circumcenter of the triangle.

26. In Ex. 25, prove that the diagonal plane $ACGE$ is perpendicular to the plane of $\triangle AFH$.

27. Each lateral edge of a prism is 12 in. and is inclined at an angle of 60° to the planes of the bases. How long is the altitude?

28. In Ex. 27, find the number of degrees in the acute dihedral angle formed by extending the plane of a right section to meet the plane of one base.

29. In Ex. 28, if the area of one base of the prism is 20, what is the area of the right section? (See § 80.)

30. In any parallelepiped, if a line passing through the center $O$ of the solid is terminated by two opposite faces of the solid, then this line is bisected by the point $O$. Prove.

31. In any parallelepiped connect the mid-points of two opposite edges. In like manner connect the mid-points of each of the remaining pairs of opposite edges. Prove that these lines just drawn all meet one another at the center of the parallelepiped and are bisected by the center. How many of these lines are there?

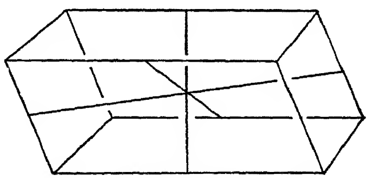


FIG. 112

32. $ABCD$ is a parallelogram in which $DB$ and $AC$ are diagonals. Using the Law of Cosines prove that

$$AC^2 + DB^2 = AB^2 + BC^2 + CD^2 + DA^2.$$

33. In any parallelepiped prove that the sum of the squares of the four diagonals equals the sum of the squares of the twelve edges. (Cf. Ex. 32.)

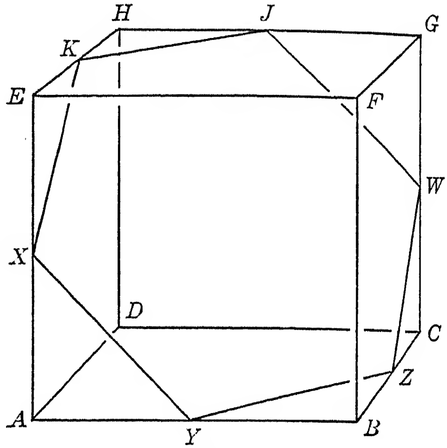



FIG. 113

**34.** $ABCD$–$EFGH$ is a cube. $X, Y, Z, W, J, K$ are respectively the mid-points of $EA$, $AB$, $BC$, $CG$, $GH$, $HE$. Prove that $X, Y, Z, W, J, K$ are coplanar, and that $XYZWJK$ is a regular hexagon.

**35.** In Ex. 34, show that diagonal $FD$ is perpendicular to the plane of $XYZWJK$ at the geometric center of that polygon.

**36.** Given a right prism $ABC$–$DEF$. Show how to construct a prism which shall be congruent (identically the same as) to the given prism. Prove the construction to be correct by showing that one prism can be made to coincide with the other.

# *Chapter Six*

## CYLINDERS

**94. Cylindric Surface.** Let $c$ be any plane curve, closed or open. Let $x$ be any straight line touching $c$ but not coplanar with $c$. Let $x$ move so that it is always in contact with $c$ and so that it remains parallel to its original position. The surface traced by $x$ is called a *cylindric surface.*

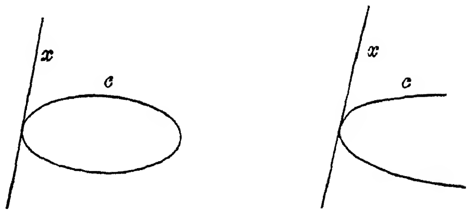



FIG. 114

Line $x$ is the *generatrix* of the surface. Curve $c$ is the *directrix*. An *element* of the cylindric surface is the line $x$ in any one of its innumerable positions.

(In all the following work it will be assumed that the directrix $c$ is a closed curve.)

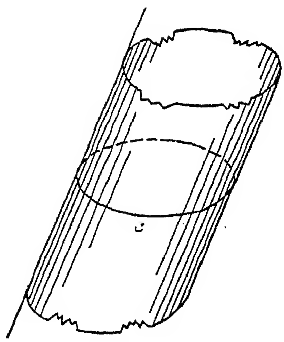


FIG. 115

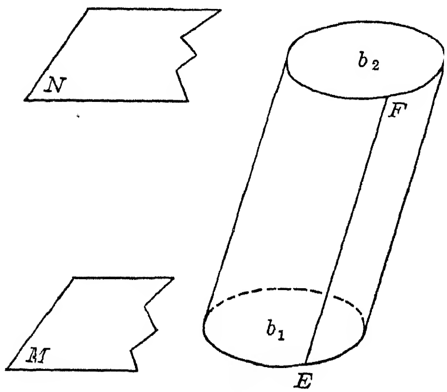



FIG. 116

**95. Cylinder.** If two parallel planes $M$ and $N$ cut all the elements of a cylindric surface, the solid bounded by $M$, $N$ and the cylindric surface is called a *cylinder* (Fig. 116).

**96. Parts of a Cylinder.** (Cf. Fig. 116.)

An *element of a cylinder* is that portion of an element of the cylindric surface which is included between $M$ and $N$ ($EF$ in Fig. 116).

The *bases* of a cylinder are the plane sections $b_1$ and $b_2$ formed on $M$ and $N$, respectively.

The *basal edges* are the circumferences of the bases.

*Corresponding points* of the basal edges are points which are the extremities of an element (points $E$ and $F$ in Fig. 116).

The *lateral surface* or *cylinder wall* is that portion of the cylindric surface which is included between the planes of the bases.

The *altitude* of a cylinder is the perpendicular distance between the planes of the bases.

**97. Right Section of a Cylinder.** If a plane cuts all the elements of a cylinder perpendicularly, the plane section thus formed is called a *right section* of the cylinder (section $s$ in Fig. 117).

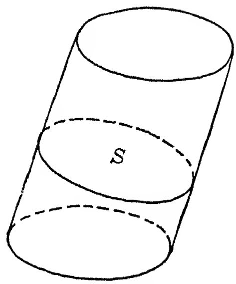



Fig. 117

**98.** Theorem 29.

The bases of any cylinder are congruent.

*Given:* Cylinder with bases $b$ and $b'$; circumferences of bases are $c$ and $c'$.

*Prove:* $b \cong b'$.

We are to prove the theorem by showing that $b$ can be moved down and made to coincide with $b'$. To do this we must show that all the points of $c$ can be made to coincide with the corresponding points of $c'$.

1) Choose any three points $A, B, C$ on $c$. Draw elements $AA'$, $BB'$, $CC'$.
2) $\therefore$ $A'$, $B'$, $C'$ correspond to $A$, $B$, $C$.
3) Draw $AB$, $BC$, $CA$, $A'B'$, $B'C'$, $C'A'$.
4) Prove $\triangle ABC \cong \triangle A'B'C'$. Then $A$, $B$, $C$ can be made to coincide with $A'$, $B'$, $C'$.
5) At the same time and by the same method we can prove that any and all other points of $c$ will coincide with the corresponding points of $c'$.

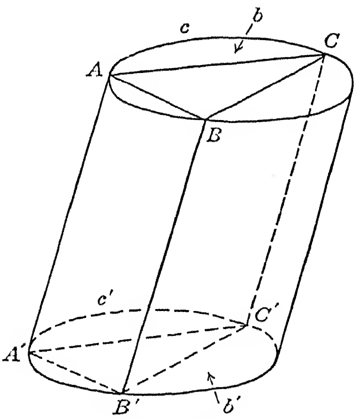



Fig. 118

6) The planes of $b$ and $b'$ and the curves $c$ and $c'$ can be made to coincide. Hence, $b \cong b'$.

**99. General Classification of Cylinders.**

A cylinder is $\left\{\begin{matrix}\text{circular}\\ \text{elliptic}\end{matrix}\right\}$ if its bases are $\left\{\begin{matrix}\text{circles}\\ \text{ellipses}\end{matrix}\right\}$, etc.

A *right cylinder* is one whose elements are perpendicular to the planes of the bases.

An *oblique cylinder* is one which is not a right cylinder.

A *right circular cylinder* or *cylinder of revolution* is a right cylinder whose bases are circles.

**100. Parts of a Right Circular Cylinder.** The *axis* of a right circular cylinder is the line joining the centers of the circles which form the bases. (The term *axis* is similarly applied to any type of circular cylinder.)

An *axial section* of a right circular cylinder (or of any circular cylinder) is a plane section which contains the axis.

The *radius* of a right circular cylinder is the radius of either base.

**101. Similar Cylinders of Revolution.** If a rectangle is rotated through 360° about one of its sides, the rectangle will generate a cylinder of revolution. If similar rectangles are rotated about corresponding sides, the cylinders of revolution thus generated are called *similar cylinders of revolution*.

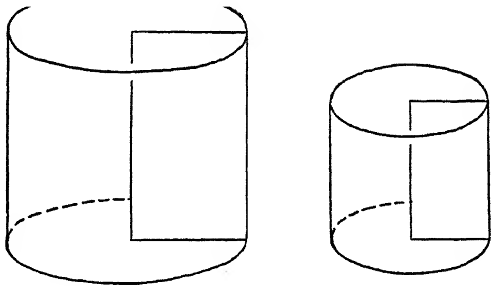


FIG. 119

**102. Tangent Plane.** If a plane touches a cylindric surface so that the two have one and only one element in common, the plane and the cylindric surface are said to be *tangent* to each other.

**103. Circumscribed Prism.** If the bases of a prism are respectively in the planes of the bases of a cylinder, and if the lateral faces of the prism are each tangent to the cylinder wall, the prism is said to be *circumscribed* about the cylinder, — or the cylinder is said to be *inscribed* in the prism (Fig. 120).

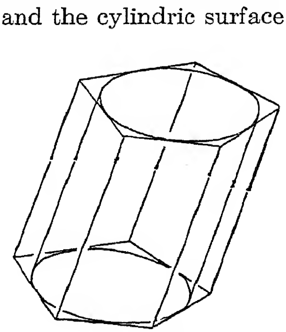


FIG. 120

**104. Inscribed Prism.** If the bases of a prism are respectively in the planes of the bases of a cylinder, and if the lateral edges of the prism are each elements of the cylinder, the prism is said to be *inscribed* in the cylinder, — or the cylinder is said to be *circumscribed* about the prism (Fig. 121).

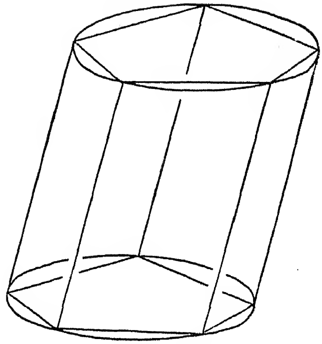


Fig. 121

## EXERCISES

*Group Nine*

1. If a plane containing an element of any cylinder intersects the cylinder, its second intersection with the cylinder wall is an element. (Let plane $S$, containing element $AB$, intersect the cylinder wall for the second time in a line $x$. Through $C$ draw element $CD$. Show that both $CD$ and $x$ lie in $S$ and the cylinder wall at the same time, and hence that $x$ and $CD$ coincide since there can be but one intersection of the plane and the cylinder wall in this region of the wall.)

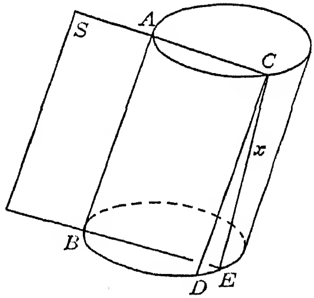



Fig. 122

2. A section of a cylinder made by a plane containing an element is a parallelogram.

3. A section of a right cylinder made by a plane containing an element is a rectangle.

4. In a right cylinder the altitude equals any element.

5. Any axial section of a cylinder of revolution is a rectangle.

6. Sections of a cylinder made by two parallel planes cutting all the elements are congruent.

7. Any two right sections of a given cylinder are congruent.

8. Any right section of a cylinder of revolution is a circle.

9. The axis of a circular cylinder equals any element, is parallel to all the elements, and passes through the centers of all sections parallel to the bases.

10. If a cylinder is inscribed in a prism, the lateral edges of the prism are parallel to the elements of the cylinder.

**11.** If a prism is inscribed in a cylinder, the lateral faces of the prism are each parallel to the elements of the cylinder.

**12.** A regular prism can be inscribed in or circumscribed about a given cylinder of revolution, and the line joining the centers of the bases of the prism must coincide with the axis of the cylinder.

**13.** Find the radii of the cylinders which can be inscribed in and circumscribed about a cube one edge of which is 6 in.

**14.** The altitude of a right circular cylinder is 10 in. and its radius is 3 in. A right prism with equilateral triangles as bases is inscribed in the cylinder. Find the sum of the areas of all the faces of the prism.

**15.** Do Ex. 14, assuming that the prism is circumscribed about the cylinder.

**16.** What is the locus of points which are 2 in. from a given line?

**17.** $A$ and $B$ are points which are respectively 4 in. and 5 in. from a line $x$. What is the locus of points which are at the same time 2 in. from $x$ and equidistant from $A$ and $B$?

**18.** A line $x$ lies between the faces of a dh $\angle M\text{–}h\text{–}N$; $x$ is parallel to $h$. What is the locus of points which are at the same time $d$ in. from $x$ and equidistant from $M$ and $N$?

**19.** A line $x$ is perpendicular to a plane $M$. What is the locus of points which are 7 in. from $x$ and 4 in. from $M$?

**20.** Do Ex. 19, assuming that $x$ is parallel to $M$ and 1 in. away from $M$.

**21.** In a rectangle $ABCD$, $AB = 6$ in., $BC = 4$ in. A cylinder of revolution is generated by rotating $ABCD$ about $BC$. What is the area of the axial section? What is the area of the axial section if the cylinder is generated by rotating $ABCD$ about $AB$?

# *Chapter Seven*

## PRISMS AND CYLINDERS: AREAS AND VOLUMES

**105. Areas.** The *lateral area* of a prism is the sum of the areas of its lateral faces. The *lateral area* of a cylinder is the area of the cylinder wall. The *total area*, either of a prism or cylinder, is the sum of the lateral area and the areas of the bases.

**106. Volume.** The volume of any solid is the amount of space included by the bounding surfaces.

**107.** THEOREM 30.

> The lateral area of a prism is the product of the perimeter of a right section and a lateral edge. $S = t \cdot e.$

*Given:* Any prism.
$e$ = lateral edge;
$t$ = perimeter of right section;
$S$ = lateral area.

*Prove:* $S = t \cdot e.$

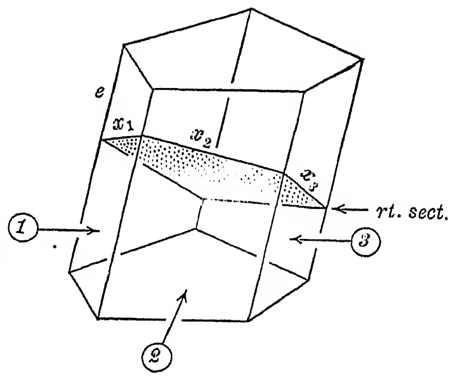


FIG. 123

1) Each of the lateral faces, 1, 2, 3, etc., is a parallelogram; and the lateral area is the sum of the areas of these parallelograms. Also, $x_1$, $x_2$, $x_3$, etc., are each perpendicular to the lateral edges which they meet.

2) $\therefore$ areas of parallelograms are: $k_1 = x_1 \cdot e$
$k_2 = x_2 \cdot e$
$k_3 = x_3 \cdot e$; etc.

3) Adding: $(k_1 + k_2 + k_3 + \cdot\cdot\cdot) = (x_1 + x_2 + x_3 + \cdot\cdot\cdot)e$

4) Or: $(\quad S \quad) = (\quad t \quad)e$

**108.** ASSUMPTION. If a prism is inscribed in or circumscribed about a cylinder, and if the number of lateral faces of the prism is caused to become infinite,

i) the lateral area and total area of the prism approach, respectively, the lateral area and total area of the cylinder as limits;

ii) the volume of the prism approaches the volume of the cylinder as a limit.

**109. Analogy between Prisms and Cylinders.** § 108 states that if a prism is inscribed in or circumscribed about a cylinder and if the number of lateral faces of the prism is greatly increased, the prism becomes more and more like the cylinder. The greater the number of lateral faces becomes, the smaller each face becomes, and the more closely the prism approximates the cylinder both in appearance and in measurement. The content of § 108 is comparable with that of Postulate 5 which has to do with the behavior of a polygon inscribed in or circumscribed about a circle when the number of sides of the polygon is increased indefinitely.

It will soon be seen that the formulas for areas and volumes of cylinders are out-growths of the corresponding formulas for prisms. The results for cylinders will be achieved by the use of § 108.

It is valuable to bear in mind constantly the analogy between prisms and cylinders. In general, if a theorem is valid for a prism it is likewise valid for a cylinder, — provided that the theorem is not concerned with the number of faces or with the separate faces of the prism. A cylinder is the limiting form of an inscribed or a circumscribed prism as the number of lateral faces of the prism is increased indefinitely.

**110.** Theorem 31.

The lateral area of a cylinder is the product of the perimeter of a right section and an element. $S = t \cdot e$.

*Given:* Any cylinder.
$e$ = element;
$t$ = per. of rt. sect.;
$S$ = lat. area.

*Prove:* $S = t \cdot e$.

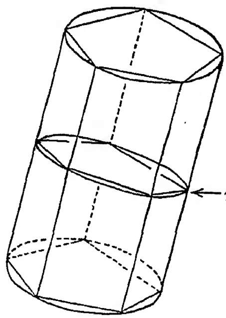



Fig. 124

1) Inscribe a prism in the cylinder.
For this prism: $e$ = lat. edge.
Let $t'$ = per. rt. sect.;
and $S'$ = lat. area.

2) $\therefore S' = t' \cdot e$ (§ 107).

3) Let the number of lateral faces of the prism become infinite. * Then $t' \rightarrow t$ (Post. 5). Hence, $t' \cdot e \rightarrow t \cdot e$, since $e$ remains constant. Also, $S' \rightarrow S$ (§ 108).

4) The two variables $S'$ and $t' \cdot e$ are always equal.

5) $\therefore$ their limits, $S$ and $t \cdot e$ must be equal (Ref. 91).

6) That is, $S = t \cdot e$.

* The arrow signifies "approaches as a limit."

**111. Corollary A** (Th. 31).

In a cylinder of revolution with radius $r$ and altitude $h$:

Lateral Area: $S = 2\pi rh$.
Total Area: $T = 2\pi rh + 2\pi r^2$.

**112. Corollary B** (Th. 31).

The lateral areas or the total areas of two similar cylinders of revolution are to each other as the squares of their respective elements, altitudes, radii, diameters.

1) $S_1 = 2\pi r_1 h_1$
$S_2 = 2\pi r_2 h_2$

2) $\therefore \dfrac{S_1}{S_2} = \dfrac{2\pi r_1 h_1}{2\pi r_2 h_2} = \dfrac{r_1 h_2}{r_2 h_1} \quad \dfrac{r_1}{r_2}\Big)\Big(\dfrac{h_1}{h_2}\Big).$

3) But $\dfrac{r_1}{r_2} = \dfrac{h_1}{h_2}$. Why?

Complete the proof.

## EXERCISES

*Group Ten*

**1.** The lateral area of a right prism is the product of the altitude and the perimeter of a base. Prove.

**2.** A right section of a prism is a regular hexagon 3 in. on a side; the lateral edge of the prism is 10 in. Find the lateral area.

**3.** The radius of a right circular cylinder is 8 in. and the altitude is 5 in. Find the lateral area and the total area.

**4.** If the wall of a right circular cylinder is unrolled from the solid and spread out upon a plane what shape will it then have?

**5.** A rectangle $ABDC$ in which $AB = 4$ in. and $AC = 12$ in. is rolled so as to become the wall of a right circular cylinder, one element of which is 4 in. What is the radius and total area of this cylinder? (Assume that edge $AB$ just meets edge $CD$ when the cylinder is formed.)

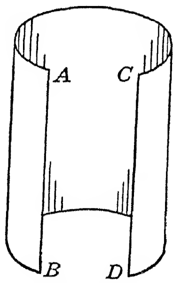



FIG. 125

**6.** Each edge of a regular hexagonal prism is 6 in. Find the total area.

7. In Ex. 6 find the lateral areas of the circular cylinders which are respectively inscribed in and circumscribed about the prism.

8. $ABCD$–$EFGH$ is a parallelepiped. $ABCD$ is a rectangle. $AB = 10\sqrt{3}$ in., $DA = 6$ in., $AE = 8$ in. Faces $ABFE$ and $DCGH$ are perpendicular to the planes of the bases. $\angle EAB = 60°$. Find the lateral area and the total area.

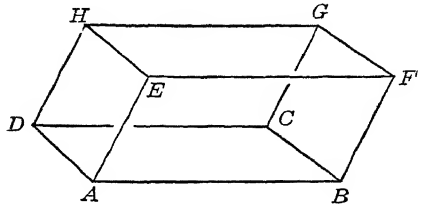



FIG. 126

9. In Ex. 8 what is the angle of inclination of a right section to the plane of a base?

10. A piece of drain pipe, cylindrical in shape, is 3 ft. long. The wall of the pipe is $\frac{3}{4}$ in. thick; the inner radius of the pipe is 8 in. Find the total area of the piece of pipe: inside, outside, edges. (Assume pipe to be a *right* cylinder in shape.)

11. The radius and altitude of a right circular cylinder are each 12 in. Find the lateral area and the total area of the regular triangular prisms which are respectively inscribed in and circumscribed about the cylinder.

12. Two rectangles whose dimensions are respectively 3 by 4 and 12 by 16 are revolved about corresponding sides as axes. What is the lateral area of each cylinder if the rectangles are revolved about their smaller sides? What are the lateral areas of the cylinders formed if the rectangles are revolved about their longer sides?

13. $C$ and $D$ are two similar cylinders of revolution. In $C$ the altitude is 8 in. and the radius 3 in. In $D$ the altitude is 6 in. What is the radius of $D$?

14. In Ex. 13 what is the ratio of the lateral area of the smaller cylinder to that of the larger?

15. $C$ and $D$ are two similar cylinders of revolution. The diameter of $C$ is 10 in. and the radius of $D$ is 15 in. What is the ratio of the lateral area of the larger to that of the smaller?

16. The altitude of one of two similar cylinders of revolution is four times that of the other. What is the ratio of the lateral area of the first to that of the second? Compare their total areas, also.

17. The lateral area of one of two similar cylinders of revolution is nine times that of the other. What is the ratio of the radii, — smaller to larger?

18. The lateral area of one of two similar cylinders of revolution is $k$ times that of the second. What is the ratio of the altitude of the first to the altitude of the second?

19. The sum of the lateral areas of two similar cylinders of revolution is 200 sq. in. The radius of the larger is three times that of the smaller. Find the lateral area of each.

20. The sum of the altitudes of two similar cylinders of revolution is 40 in. The lateral area of the smaller is $\frac{1}{16}$ that of the larger. Find the altitude of each.

21. The dimensions of a rectangle are $x$ in. by $y$ in. The rectangle is rotated about its $x$-inch side to generate a right circular cylinder. It is then rotated about its $y$-inch side instead. Compare the lateral areas of the two different cylinders thus formed. Compare the total areas.

**22.** The radius of a right circular cylinder is 10 in. A plane parallel to the axis of the cylinder and 6 in. from the axis cuts the cylinder. If the altitude of the cylinder is 12 in., find the area of the plane section thus formed.

**23.** The radius of a right circular cylinder is 5 in. Through an external point 5 in. from the cylinder wall two planes are drawn tangent to the cylinder. How large is the dihedral angle formed by these planes?

**113. Units of Measurement.** In order to measure a distance we use a unit of length, namely, an arbitrarily chosen line-segment $u$ (inch, foot, centimeter, etc.). The measurement of this distance is the number of times the chosen unit $u$ is contained in the given distance.

Similarly, in order to measure an area, we commonly use a square one side of which is a standard length unit (square inch, square foot, square centimeter, etc.). The measurement of the given area is the number of times this unit square is contained in the given area.

Finally, in order to measure a volume, we ordinarily use a unit cube each edge of which is a standard length unit and each face of which, correspondingly, is a standard area unit (cubic inch, cubic foot, cubic centimeter, etc.). The measurement of the volume is the number of unit cubes contained in the given volume.

The measurement of a length, area, or volume may be any sort of positive, real number, — rational or irrational.

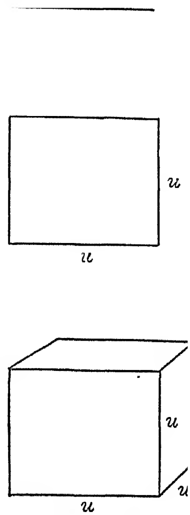



FIG. 127

**114.** THEOREM 32.

> The volume of a rectangular solid is the product of the area of its base by its altitude, — or what is the same thing, the product of its three dimensions.

A rigorous proof of this theorem will not be given here. In fact we shall accept the truth of it without any proof at all.

In case each dimension is a whole number the truth of the theorem is apparent. In the figure, for example, the number of unit cubes contained in the solid is 140 since there are 5 horizontal layers of cubes and since each layer contains 4 rows of cubes with 7 cubes to a row.

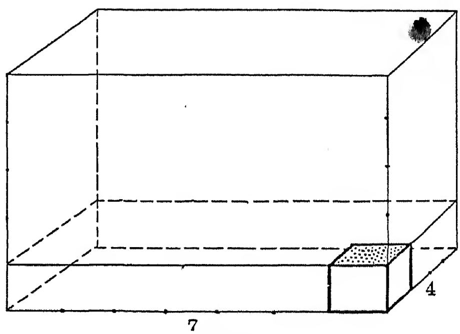



FIG. 128

In a rigorous proof, however, we

should have to consider the case where one or more dimensions is a fraction or mixed number, and also the case where one or more dimensions is an irrational number such as $\sqrt{3}$, $5\sqrt{7}$, etc.

**115. Corollary A** (Th. 32).

The volume of a right prism having right triangles as bases is the product of a base and the altitude.

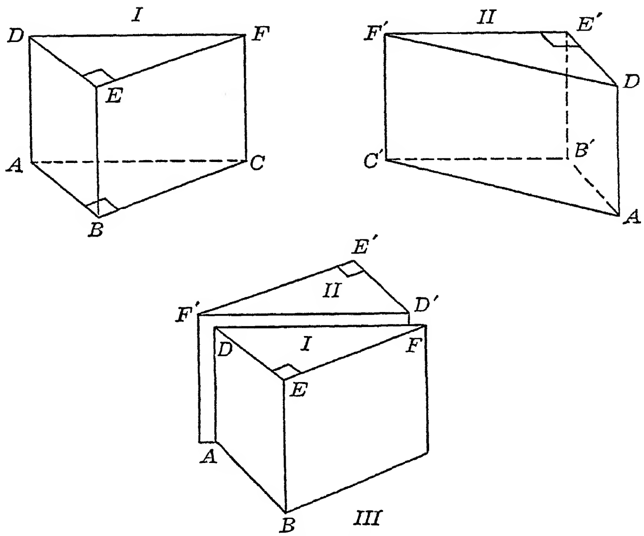


FIG. 129

*Given:* rt prism $ABC\text{–}DEF$. $\triangle ABC$, $\triangle DEF$ rt ⊿s. Area $\triangle ABC = b$; altitude $= h$. $V =$ volume.

*Prove:* $V = b \cdot h$.

1) Construct a second prism II: $A'B'C'\text{–}D'E'F'$ which is identically like the given prism I. Points $A'$, $B'$, $C'$, $D'$, $E'$, $F'$ correspond to $A$, $B$, $C$, $D$, $E$, $F$, respectively. (Cf. Ex. 36, Group Eight.)
2) Place II against I so that face $A'D'F'C'$ coincides with its equal $ADFC$, but with $A'D'$ coinciding with $CF$ and $C'F'$ coinciding with $AD$. Show that the composite solid III thus formed is a rectangular solid with base $2b$ and altitude $h$.
3) $\therefore\ V_{\text{III}} = 2b \cdot h$ (§ 114).
4) But $V = \frac{1}{2}V_{\text{III}}$, since I and II are identically alike.
5) $\therefore\ V = \frac{1}{2}(2b \cdot h) = b \cdot h$.

**116. Corollary B** (Th. 32).

The volume of any right prism is the product of its base and altitude.

Show that if certain planes are drawn to contain the lateral edges the prism can be resolved into several right prisms having right triangles as bases.

Apply § 115. Combine the results.

**117.** THEOREM 33.

In any prism the product of its base and altitude equals the product of the area of a right section and a lateral edge.

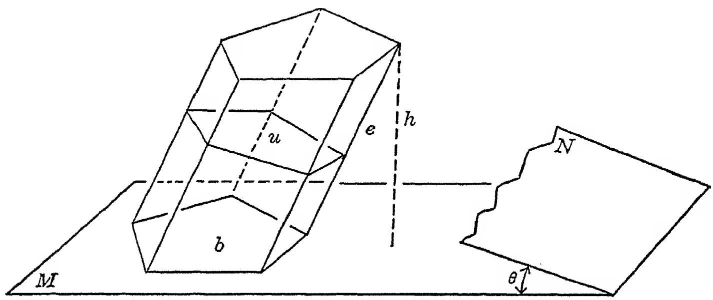



FIG. 130

*Given:* Any prism.
$b$ = area of base
$u$ = area of right section
$h$ = altitude
$e$ = lateral edge.

*Prove:* $b \cdot h = u \cdot e$.

1) Let the plane $N$ of the right section intersect plane $M$ of the base, forming a dh $\angle \theta$.
2) Thus the right section is the projection of the lower base upon the plane $N$.
3) $\therefore \quad u = b \cos \theta$. (See § 80 and Exs. 10, 11 of Group Seven.)
4) Or $\quad b = u \sec \theta$.
5) Also, $h = e \cos \theta$. Why?
6) $\therefore\ b \cdot h = u \sec \theta\, e \cos \theta$
$\qquad = u \cdot e(\sec \theta \cos \theta)$
$\qquad = u \cdot e(1)$
$\qquad = u \cdot e$.

**118.** THEOREM 34.

The volume of any given prism is the same as the volume of a right prism whose base is a right section of the given prism and whose altitude is equal to a lateral edge of the given prism.

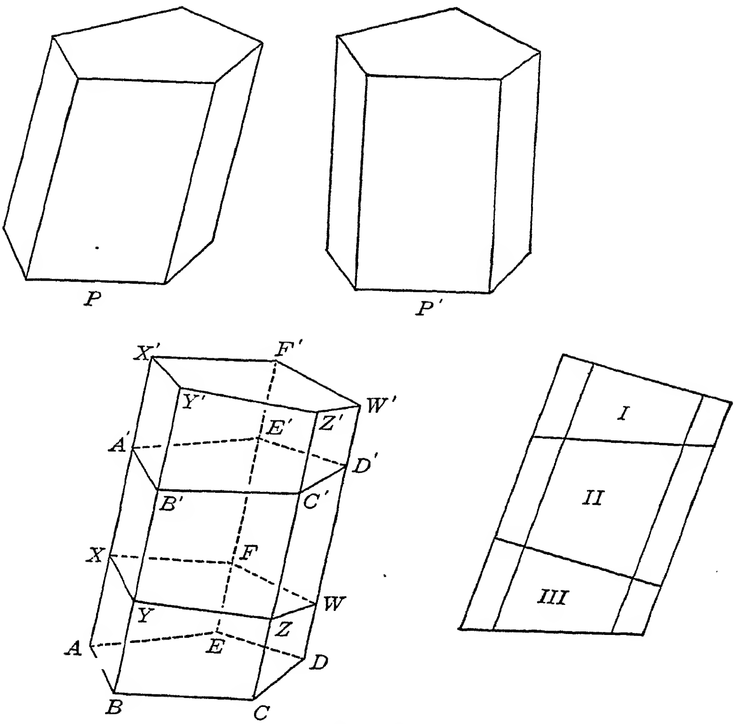



FIG. 131

*Given:* Any oblique prism $P$
Right prism $P'$
Base of $P'$ = right section of $P$
Altitude of $P'$ = a lateral edge of $P$.

*Prove:* Vol. of $P$ = vol. of $P'$.

1) Since $XYZWF$ is the same as a right section of $P$, $P'$ may be partially fitted over $P$ as shown. The resulting composite solid is then composed of three truncated prisms:

$$\text{I: } A'B'C'D'E'\text{–}X'Y'Z'W'F'$$
$$\text{II: } XYZWF\text{–}A'B'C'D'E'$$
$$\text{III: } ABCDE\text{–}XYZWF.$$

2) Show that vol. I = vol. III by showing that I and III can be made to coincide.
3) But $P = \text{III} + \text{II}$, and $P' = \text{I} + \text{II}$.
4) $\therefore P = P'$.

**119.** THEOREM 35.

The volume of any prism is the product of its base and altitude.

$$V = b \cdot h.$$

*Given:* Any prism $P$.
$b$ = area of base
$h$ = altitude
$u$ = area rt. sect.
$e$ = lat. edge
$V$ = vol.

*Prove:* $V = b \cdot h$.

1) Let $P'$ be a right prism having $u$ as base and $e$ as altitude. Let $V'$ be its volume.
2) $\therefore\ V = V'$ (§ 118).
3) Also, $V' = u \cdot e$ (§ 116).
4) But $u \cdot e = b \cdot h$ (§ 117).
5) $\therefore\ V = V' = u \cdot e = b \cdot h$
or
$V = b \cdot h$.

**120.** THEOREM 36.

The volume of any cylinder is the product of its base and altitude.

$$V = b \cdot h.$$

*Given:* Any cylinder.
$b$ = area base
$h$ = altitude
$V$ = volume

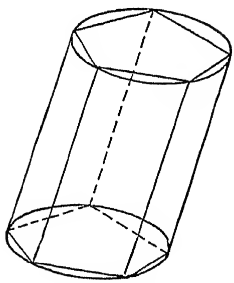


FIG. 132

*Prove:* $V = b \cdot h$.

1) Inscribe a prism in the cylinder. For this prism let $b'$ = area base, $V'$ = volume. $h$ = altitude. Why?
2) $\therefore\ V' = b' \cdot h$. (§ 119)
3) Let the number of lateral faces of the prism become infinite. Then $b' \rightarrow b$ (Post. 5). $\therefore\ b' \cdot h \rightarrow b \cdot h$, since $h$ remains constant. Also $V' \rightarrow V$ (§ 108).
4) The two variables $V'$ and $b' \cdot h$ are always equal. $\therefore$ their limits $V$ and $b \cdot h$ must be equal (Ref. 91).
5) $\therefore\ V = b \cdot h$.

**121. Corollary A** (Th. 36).

In a cylinder of revolution of radius $r$ and altitude $h$: $V = \pi r^2 h$.

**122. Corollary B** (Th. 36).

> The volumes of two similar cylinders of revolution are to each other as the cubes of their respective radii, diameters, altitudes, elements.

(For general method of proof see § 112.)

**123. The Generalized Cylindric Solid** (Fig. 133). A cylindric surface has already been defined as the surface generated by a moving straight line having a fixed direction and always in contact with some fixed plane curve, the line itself not being coplanar with the curve. The moving line and the fixed plane curve, respectively, were called generatrix and directrix.

This idea may be generalized as follows. Let $f$ be any plane figure such as the sector of a circle, or a figure composed of any combination of straight line segments and arcs of curves. Let $e$ be any straight line not coplanar with $f$; let $e$ maintain a fixed direction, and let $e$ be in contact with $f$ at all times. As $e$ moves it generates a surface comparable to a cylindric surface. We may regard this surface as a *generalized cylindric surface* having $e$ as its generatrix and $f$ as its directrix.

If we form a solid by letting two parallel planes cut all the elements of one of these generalized cylindric surfaces, we obtain a solid which we may call a *generalized cylindric solid*. (See the illustrations below.)

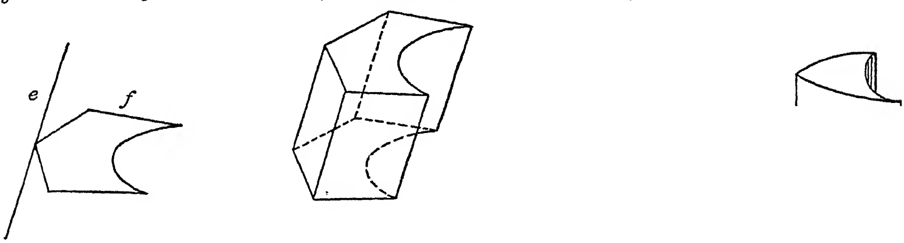



FIG. 133

Many of the properties of these generalized cylindric solids, and in particular the measurement formulas for area and volume, can be shown to be the same as those already developed for ordinary prisms and cylinders. For example, the lateral area can be found by taking the product of the perimeter of a right section and an element. The volume is the product of area of base and altitude. We shall assume these formulas to be valid without presenting any proof.

## EXERCISES

*Group Eleven*

**1.** Find the volume of a right circular cylinder if $r = 4$ in. and $h = 6$ in.

**2.** In two similar cylinders of revolution the radius of one is 3 in. and that of the other is 4 in. What is the ratio of the volume of the first to that of the second?

**3.** In two similar cylinders of revolution the altitude of the first is $k$ times the altitude of the second What is the ratio of the volume of the first to that of the second?

**4.** In two similar cylinders of revolution the volume of one is $\frac{1}{8}$ the volume of the other. In comparing the first to the second what is the ratio of the radii? Lateral areas? Total areas?

**5.** Do Ex. 4 assuming that the first volume is $k$ times the second.

**6.** The lateral areas of two similar cylinders of revolution are in the ratio $\frac{4}{9}$. What is the corresponding ratio of their volumes?

**7.** The total area of one of two similar cylinders of revolution is 6 times that of the other. What is the ratio of the volume of the larger to that of the smaller?

**8.** The sum of the volumes of two similar cylinders of revolution is 112 cu. in. Their lateral surfaces are in the ratio $\frac{1}{9}$. Find the volume of each.

**9.** In Exercise 10 of Group Ten find the number of cubic inches in the wall of the drain pipe.

**10.** A clock case is constructed in the form of a rectangular solid $ABCD\text{-}EFGH$ surmounted by half of a right circular cylinder as shown. $AB = 8$ in., $BC = 5$ in., $BF = 10$ in. Find the number of cubic inches of space within the case.

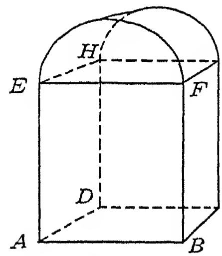



FIG. 134

**11.** A cubical block of wood, 4 in. on an edge, has a round hole 3 in. in diameter bored through it. The axis of this cylindrical hole passes through the center of the cube and is parallel to four lateral edges. Find the amount of wood left after the hole is bored.

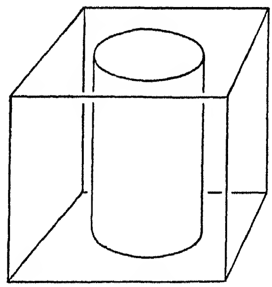


FIG. 135

**12.** The volume of a right circular cylinder is $\frac{1}{2}$ the product of its lateral surface and radius. Prove.

**13.** The number of cubic inches in the volume of a certain right circular cylinder is the same as the number of square inches in its lateral area. Find the number of inches in the radius.

14. Find the volume of the right circular cylinder which can be circumscribed about a cube one diagonal of which is $3\sqrt{3}$ in.

15. $ABCD$–$EFGH$ is an oblique parallelepiped each edge of which is 10 in. The bases $ABCD$ and $EFGH$ are squares. The vertex $E$ is situated so that a perpendicular from $E$ to the plane of $ABCD$ would cut that plane at the center of square $ABCD$. Find the volume of the solid.

16. The bases of a right cylindric solid are each 135° sectors of circles with 4 in. radius. The altitude of the solid is 12 in. Find the volume, lateral area, and total area.

17. In a right circular cylinder the altitude is 10 in. and the radius is 6 in. A plane parallel to the axis and 3 in. from the axis cuts the cylinder as shown. Find the volumes of the two solids into which this plane separates the given cylinder.

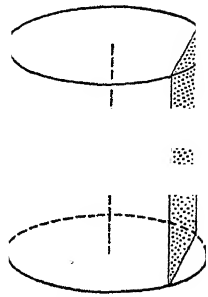


FIG. 136

18. In a figure like that of the preceding exercise, if $a$ is the area of the base of one of the pieces and $b$ the area of the base of the other; and if $v$ is the volume of the first piece and $w$ the volume of the second, prove: $v = \frac{a}{b} w$

19. In a right prism $ABC$–$DEF$, $CA = 15$ in., $CB = 20$ in., $\angle ACB = 90°$, $CF = 30$ in. A plane containing lateral edge $CF$ is drawn perpendicular to lateral face $ADEB$ cutting it in $HK$. Find the volumes of prisms $AHC$–$DKF$ and $CHB$–$FKE$.

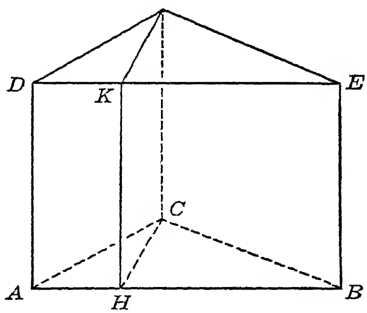



FIG. 137

20. A block of lead has the shape of a rectangular solid whose dimensions are $\frac{1}{2}$ in. by $\frac{3}{4}$ in. by 1 in. This piece of lead is to be melted and recast into the form of a regular triangular prism each basal edge of which is to be $\frac{1}{2}$ in. What must be the altitude of this prism?

**21.** The altitude of a right circular cylinder equals its diameter. The volume is $128\pi$ cu. in. Find the altitude and diameter.

**22.** The altitude of a right circular cylinder is 7 in. and the total area is $88\pi$ sq. in. Find the radius.

**23.** If $h$ is the altitude of a right circular cylinder and $r$ the radius, find the ratio of $h$ to $r$ if the lateral area is known to be 3 times the sum of the areas of the bases.

**24.** Do Ex. 23, finding the ratio $h/r$, assuming that the total area is 4 times the lateral area

**25.** Two planes are perpendicular to each other, meeting in line $AB$. A circular cylindric surface is tangent to these planes at $EF$ and $CD$, respectively. Assume that the planes $ACE$ and $BDF$ are perpendicular to $AB$, and that $CD$ and $EF$ are each parallel to $AB$. $AB = 8$ in., $AC = 5$ in. Find the volume and lateral area of the solid.

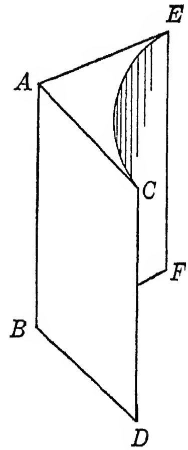



FIG. 138

**26.** A tank built in the form of a right circular cylinder of radius 10 ft. and altitude 40 ft. rests on an element. The tank is partially filled with fuel oil, the greatest depth of the oil being 4 ft. If the tank were to be raised up and made to rest upon one of its circular bases, how deep would the oil then be? Assume, of course, that the tank sets upon level ground.

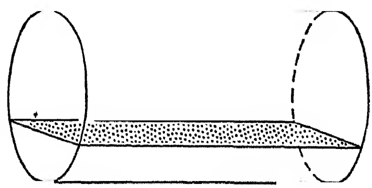


FIG. 139

**27.** A container partially filled with water is constructed in the form of a rectangular solid. The dimensions of the base are 6 in. by 8 in. A block of iron in the shape of a regular hexagonal prism with a basal edge of 2 in. is dropped into the water. When the iron is completely submerged it is found that the water level has risen exactly 1 in. Assuming that the container is level find the altitude of the iron prism. Leave answer in simplest radical form.

# Chapter Eight

## PYRAMIDS

**124. Pyramidal Surface.** Let $d$ be any polygon. Let $V$ be a fixed point not coplanar with $d$. Let $g$ be a straight line through $V$ and touching $d$ (Fig. 140). If $g$ moves, always touching $d$, it will generate a *pyramidal surface*. Obviously

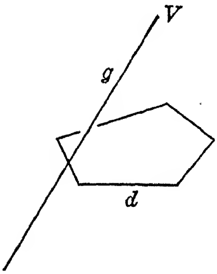



FIG. 140

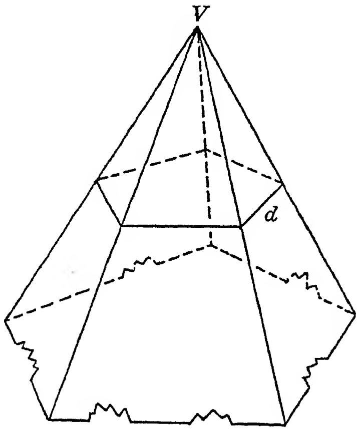



FIG. 141

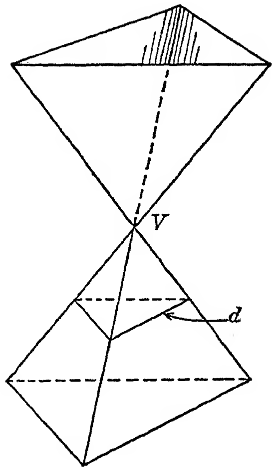



FIG. 142

a pyramidal surface is composed of a series of plane surfaces having one point, $V$, in common. (See Fig. 141.)

$V$ is the *vertex* of the surface.

$g$ is the *generatrix*.

$d$ is the *directrix*.

An *element* of a pyramidal surface is the generatrix in any one of its innumerable positions.

If the generatrix protrudes through $V$ as it moves it generates two surfaces simultaneously (Fig. 142). The two surfaces thus generated constitute what is called a pyramidal surface of *two nappes*. The surface above $V$ is the *upper nappe;* the lower surface is the *lower nappe*.

**125. Pyramid.** If a plane $M$ not containing the vertex $V$ of a pyramidal surface of one nappe is passed completely through the surface, the solid bounded by $M$ and the pyramidal surface is called a *pyramid* (Fig. 143).

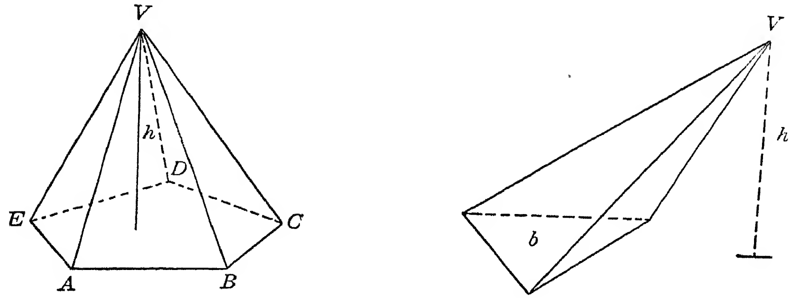



FIG. 143

**126. Parts of a Pyramid.** (Cf. Fig. 143.)

*Vertex:* the point $V$.

*Base:* the polygon or triangle determined by the intersection of $M$ with the pyramidal surface ($ABCDE$ in the figure).

*Altitude:* the perpendicular distance from vertex to plane of base ($h$).

*Basal Edges:* $AB$, $BC$, $CD$, etc.

*Lateral Edges:* $VA$, $VB$, $VC$, etc.

*Lateral Faces:* the triangles $VAB$, $VBC$, $VCD$, etc.

*Lateral Area:* the sum of the areas of the lateral faces.

*Total Area:* the area of the base plus the lateral area.

**127. Classification of Pyramids.**

| | | | | |
|---|---|---|---|---|
| | triangular | | triangle | |
| A pyramid is | quadrangular | if its base is a | quadrilateral | etc. |
| | pentagonal | | pentagon | |

A triangular pyramid is also called a *tetrahedron*. If all the edges of a tetrahedron are equal, the solid is a *regular tetrahedron*.

**128. Regular Pyramid** (Fig. 144). A *regular pyramid* or *right pyramid* is one whose base is a regular polygon or triangle, and whose vertex is directly over the geometric center of the base.

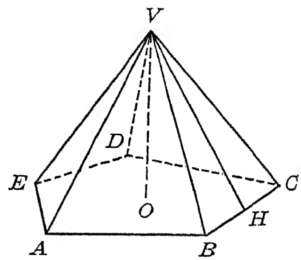



FIG. 144

**129. Slant Height.** It follows at once that the lateral edges of a regular pyramid are equal and hence that the lateral faces are congruent isosceles tri-

angles. If in each lateral face of a regular pyramid a perpendicular is drawn from the vertex to the basal edge, then all these perpendiculars must be equal. Any one of these perpendiculars is called the *slant height* of the pyramid ($VH$ in the figure). Only a regular pyramid can possess a slant height.

**130.** THEOREM 37.

If a plane parallel to the base of a pyramid cuts all the lateral edges, this plane divides the altitude and lateral edges proportionally; and the section formed is similar to the base of the pyramid.

(Proof left to student.)

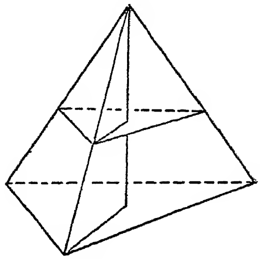


FIG. 145

**131. Corollary A** (Th. 37).

If $k$ is the area of a plane section of a pyramid parallel to the base $b$; if the distance of section $k$ from the vertex is $d$, and if the altitude of the pyramid is $h$; then:

$$k = \left(\frac{d}{h}\right)^2 b.$$

**132. Frustum of a Pyramid.** If a plane $S$ parallel to $M$, the plane of the base of a pyramid, cuts all the lateral edges, that portion of the pyramid included between $S$ and $M$ is a *frustum* of the given pyramid (Fig. 146).

The sections on $S$ and $M$ are the *bases* of the frustum ($DEF$ and $ABC$ in figure).

The *lateral edges*, *lateral faces*, and *altitude* are respectively the portions of the lateral edges, lateral faces, and altitude of the pyramid which are included between $S$ and $M$. Obviously each lateral face of a frustum is a trapezoid.

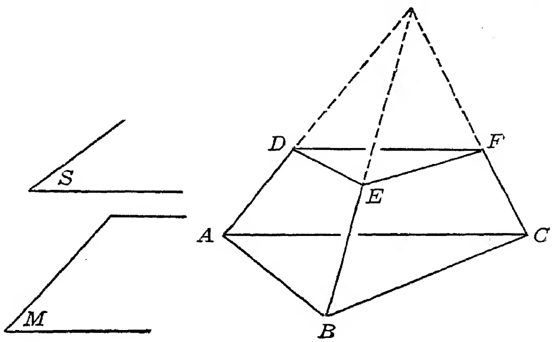



FIG. 146

It can be shown readily that the lateral edges of a frustum of a *regular* pyramid are equal, and hence that the lateral faces are congruent isosceles trapezoids. The *slant height* of a

frustum of a *regular* pyramid is that portion of the slant height of the pyramid included between the two bases of the frustum. The slant height of a frustum of a regular pyramid is the altitude of any one of the trapezoidal faces.

## EXERCISES

### *Group Twelve*

**1.** $V\text{–}ABC$ is a triangular pyramid. $VD = \frac{2}{3}VA$. Section $s$ is parallel to the base $b$. If $b = 75$ sq. in., find the area of $s$.

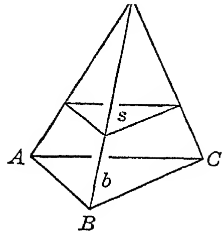



FIG. 147

**2.** The area of the base of a pyramid is 40 sq. in. A plane cuts all the lateral edges, is parallel to the base, and bisects the altitude. Find the area of the section parallel to the base.

**3.** The altitude of a pyramid is 24 in. The base is a square 6 in. on a side. How far from the base must a plane parallel to the base be drawn in order to cut the pyramid and make a section whose area is 4 sq. in.?

**4.** The altitude of a pyramid is 12 in. $s$ is a section parallel to the base $b$. $s = 8$ sq. in., $b = 18$ sq. in. Find the lengths of the segments into which this parallel section divides the altitude of the pyramid.

**5.** $ABCD\text{–}EFGH$ is a frustum of a regular square pyramid. $AB = 14$ in., $EF = 8$ in. The slant height is 5 in. Find the length of the altitude.

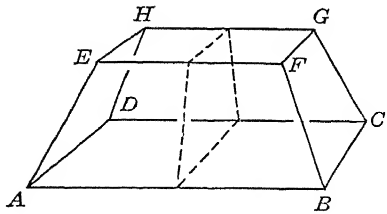



FIG. 148

**6.** In Ex. 5 find the length of each lateral edge.

**7.** In Ex. 5 find the number of degrees in the acute dihedral angle which lateral face $FBCG$ makes with the plane of base $ABCD$.

**8.** $ABC\text{–}DEF$ is a frustum of a tetrahedron. $AB = 6$ in., $BC = 8$ in., $CA = 10$ in., $DE = 3$ in. $D$ and $A$, $E$ and $B$, $F$ and $C$ are corresponding points of the two bases. Find the perimeter and area of the plane section parallel to the bases and mid-way between them. (This section is called the *mid-section.*)

9. $V$–$ABC$ is a tetrahedron. $D$ is the mid-point of $BC$. $AB = AC = VB = VC = 26$ in. $VA = 15$ in. $BC = 20$ in. Find the area of the plane section $VAD$ (Fig. 149).

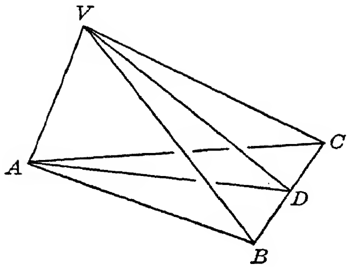



FIG. 149

10. $V$–$ABCD$ is a regular square pyramid each lateral face of which is an equilateral triangle. Find the number of degrees in the acute dihedral angle between any lateral face and the plane of the base.

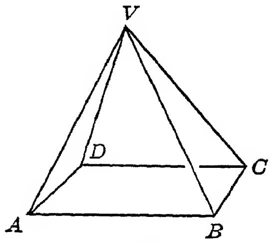



FIG. 150

11. $V$–$ABC$ is a regular tetrahedron, each edge of which is 6 in. A plane containing $BC$ cuts edge $VA$ perpendicularly at a point $D$. Find the perimeter and area of the section $BCD$.

12. $ABC$–$DEF$ is a frustum of a regular pyramid. Let $x$ = angle between lat. edge and lower base, $y$ = angle between lat. edge and lower basal edge, $z$ = angle between lat. face and lower base (Fig. 151). Prove: $y > x$ and $x < z$.

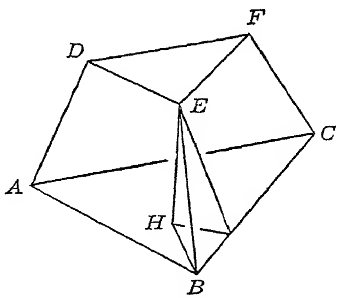



FIG. 151

13. Parallelogram $ABCD$ is the base of a pyramid $V$–$ABCD$. In lateral face $VDA$, $EF$ is parallel to $DA$, cutting $VA$ at $F$ and $VD$ at $E$. A plane $S$ containing $EF$ but not parallel to the base cuts $VB$ at $G$ and $VC$ at $H$. Prove that the section $EFGH$ is a trapezoid.

**14.** $ABC$–$DEF$ is a frustum of a pyramid. $X, Y, Z$ are respectively the mid-points of $AB$, $BC$, $CA$. Draw $DY$, $EZ$, $FX$, and prove that $DY$, $EZ$, $FX$ are concurrent (Fig. 152).

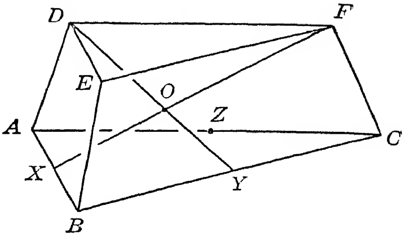



FIG. 152

**15.** $ABCD$–$EFGH$ is a frustum of a regular square pyramid. Show that the lateral edges $EA$ and $CG$ lie in one plane $S$, and that $FB$ and $HD$ lie in one plane $T$. Prove that $S$ and $T$ are perpendicular to each other and that each is perpendicular to the planes of the bases (Fig. 153).

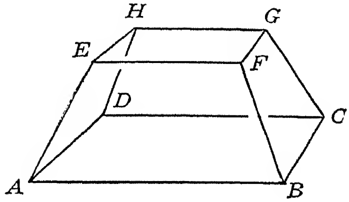



FIG. 153

**16.** In Ex. 15 show that the diagonal lines $EC$, $AG$, $FD$, $BH$ are concurrent.

**17.** $V$–$ABC$ is a tetrahedron. A plane $S$ which is parallel to the two opposite skew edges $VC$ and $AB$ cuts the pyramid so as to form the section $DEFH$. Prove that $DEFH$ is a parallelogram (Fig. 154).

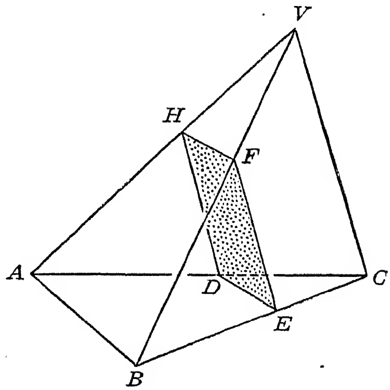



FIG 154

**18.** In Ex. 17 assume that the solid is a *regular* tetrahedron. Draw a plane $N$ containing $VC$ and bisecting edge $AB$. Prove that $N$ is perpendicular to $S$. Prove that section $DEFH$ is a rectangle.

19. $V$–$ABCD$ is any pyramid whose base is a rectangle $ABCD$ (Fig. 155). Prove:

$$VA^2 + VC^2 = VB^2 + VD^2.$$

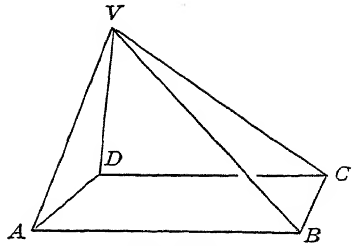



FIG. 155

20. Prove the following with respect to a regular tetrahedron (Fig. 156).
(a) All four altitudes are equal.
(b) All four altitudes are concurrent and meet one another at a point which is three-fourths the distance from any vertex to the plane of the opposite face.
(c) The point where an altitude meets a face is at the same time the centroid, circumcenter, incenter and orthocenter of that triangle.

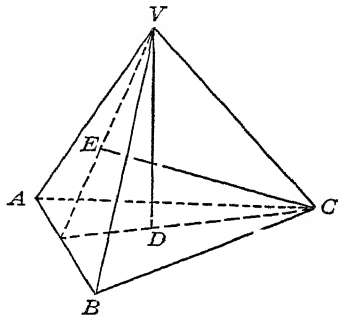



FIG. 156

**133. Median of a Tetrahedron.** In a tetrahedron a line connecting a vertex with the centroid of the opposite base is called a *median* of the tetrahedron.

21. In a regular tetrahedron all the medians are equal, and a median and an altitude are one and the same. Prove.

22. Find the altitude of a regular tetrahedron each edge of which is 6 in.; $e$ in.

23. Find the edge of a regular tetrahedron if the altitude is 12 in.; $h$ in.

24. $x, y, z, e, f, h$ are respectively the edges of a tetrahedron. Draw the six planes which bisect perpendicularly these respective edges. Prove:
(a) these planes have one and only one point in common;
(b) this point is equidistant from all four vertices.

How many of these planes are necessary to determine this point?

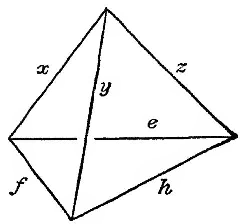



FIG. 157

**134. Circumcenter of a Tetrahedron.** The point which is equidistant from all four vertices of a tetrahedron is called the *circumcenter* of the tetrahedron. (It will be shown later that this point is the center of the sphere which can be circumscribed about the tetrahedron.)

**25.** $x$, $y$, $z$, $e$, $f$, $h$ are still the six edges of a tetrahedron. (Cf. Ex. 24 and the figure.) Draw the six planes which are respectively the bisectors of the dihedral angles $x$, $y$, $z$, $e$, $f$, $h$ of the tetrahedron. Prove:
(a) these six planes have one and only one point in common;
(b) this common point is equidistant from all four faces of the tetrahedron.
How many of these planes are necessary to determine this point?

**135. Incenter of a Tetrahedron.** The point which is equidistant from all four faces of a tetrahedron is called the *incenter* of the tetrahedron. (It will be shown later that this point is the center of the sphere which can be inscribed in the tetrahedron.)

**26.** Prove that in the case of a *regular* tetrahedron the circumcenter and the incenter are one and the same.

# Chapter Nine

## CONES

**136. Conic Surface.** A conic surface is generated in a manner similar to that in which a pyramidal surface is generated, except that the *directrix* is some sort of *plane curve*. Any conic surface discussed here is assumed to possess some type of closed curve, $c$, for its directrix, — as for example a circle or an ellipse. (See Fig. 158.)

The terms *vertex*, *generatrix*, *directrix*, *element* are employed as in the case of a pyramidal surface.

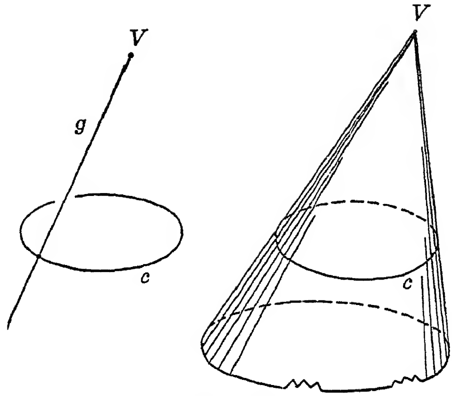



FIG. 158

**137. Conic Surface of Two Nappes.** The analogy to the pyramidal surface of two nappes is perfect. (Cf. § 124.) A conic surface of two nappes is formed if the generatrix protrudes through the vertex as it sweeps out the conic surface. The upper surface is the *upper nappe*, the lower surface is the *lower nappe* (Fig. 159).

**138. Cone.** If a plane $M$, not containing the vertex $V$ of a conic surface (of one nappe), is made to cut all the elements of the surface, the solid bounded by $M$ and the conic surface is called a *cone*.

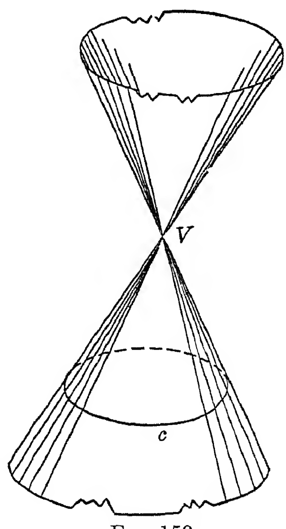



FIG. 159

**139. Parts of a Cone.** (See Fig. 160.)

*Vertex:* point $V$.

*Base:* the section determined upon $M$ by the conic surface (curved figure $b$ in Fig. 160).

*Basal Edge:* the circumference of $b$.

*Altitude:* the perpendicular distance from vertex to plane of base ($h$).

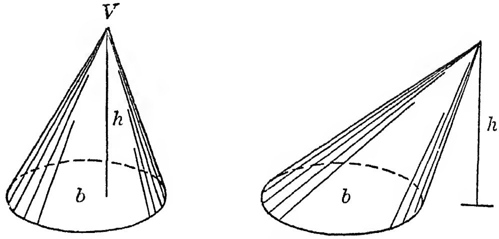



Fig. 160

*Element:* any line connecting $V$ with the basal edge.

*Lateral Surface* or *Wall:* the portion of the conic surface possessed by the cone; i.e., the curved surface.

*Lateral Area:* the area of the lateral surface.

*Total Area:* the area of base plus the lateral area.

**140. Classification of Cones.**

A cone is $\left\{ \begin{matrix} \text{circular} \\ \text{elliptical} \end{matrix} \right\}$ if its base is a $\left\{ \begin{matrix} \text{circle} \\ \text{ellipse} \end{matrix} \right\}$, etc.

The *axis* of a circular cone is the straight line connecting the vertex with the center of the base.

An *axial section* of a circular cone is a plane section which contains the axis.

The *radius* of a circular cone is the radius of its base.

(In elementary geometry we are equipped to deal only with *circular* cones. Hence all cones discussed here will be of that type.)

**141. Right Circular Cone or Cone of Revolution** (Fig. 161). If the vertex of a circular cone is directly over the center of the base, the cone is called a *right circular cone* or a *cone of revolution.*

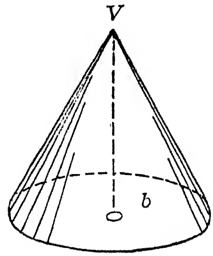



Fig. 161

Obviously all the elements of a right circular cone are equal. In this case an element is often called the *slant height* of the cone. Also, it is easily shown that the axis and altitude are one and the same.

Any axial section of a given right circular cone is congruent to any other axial section; any one of these sections is an isosceles triangle. The angle at the vertex of one of these triangles is called the *angle of the cone.* Thus, if the axial section is an equilateral triangle the angle of the cone is 60°, and so on.

**142. Similar Cones of Revolution.** If a right triangle is revolved through 360° about one of its legs it will generate a cone of revolution. If similar

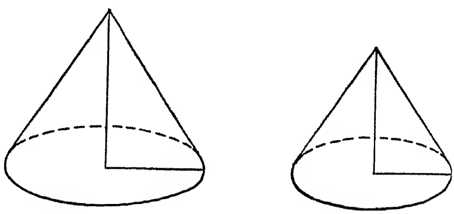


Fig. 162

right triangles are revolved about corresponding legs the cones thus generated are called *similar cones of revolution.*

**143.** THEOREM 38.

> In a circular cone a section parallel to the base divides the altitude and elements proportionally, and the section itself is a circle.

(For method of proof recall § 130 and Ex. 32 of Group Five.)

**144. Corollary A** (Th. 38).

> If $k$ is the area of a plane section of a cone parallel to the base $b$; and if the distance of section $k$ from the vertex is $d$; and if the altitude of the cone is $h$; then: $$k = \left(\frac{d}{h}\right)^2 b.$$

(Recall method of proof for § 131)

**145. Frustum of a Cone** (Fig. 163). If a plane $S$ parallel to $M$, the plane of the base of a cone, is made to cut all the elements, the solid included between $S$ and $M$ is called a *frustum* of the cone.

The terms *bases* and *altitude* are used as in frustums of pyramids. An *element* of a frustum is that portion of the element of the cone included between $S$ and $M$. By § 143 all the elements of a frustum of a right circular cone are equal; any one of these may be called the *slant height* of the frustum. The *radii* of a frustum of a circular cone are the radii of its bases. The *axis* of a frustum of a circular cone is the line connecting the centers of its bases.

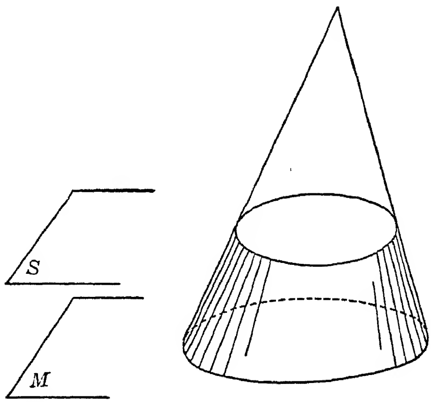


FIG. 163

## EXERCISES

*Group Thirteen*

1. Find the slant height of a cone of revolution if its altitude is 8 in. and the circumference of its base is $12\pi$ in.

2. The short leg of a 30–60–90 triangle is 4 in. The triangle is revolved about its shorter leg to generate a right circular cone. Find the altitude, element, and angle of the cone.

3. Find the area of an axial section of a cone of revolution if the angle of the cone is 90° and the slant height is $10\sqrt{2}$ in.

4. The element of a right circular cone is 20 in. and the area of its base is $144\pi$ sq. in. Find the area of the axial section.

5. The altitude of a cone of revolution is 15 in. and the radius of the base is 10 in. A plane section containing the vertex $V$ cuts the base in a line $AB$ which is 8 in. from the center $O$ of the base. Find the area of the plane section $VAB$ (Fig. 164).

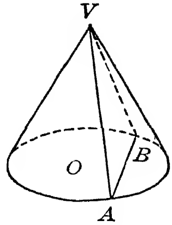


Fig. 164

6. Each element of a frustum of a right circular cone makes an angle of 60° with the plane of the lower base. The radius of the upper base is 10 in. and the altitude is 8 in. Find the radius of the lower base (Fig. 165).


Fig. 165

7. In a frustum of a right circular cone the altitude is 4 in. and the radii of the bases are respectively 2 in. and 5 in. Find the length of an element of the frustum, and the lengths of the element and altitude respectively of the entire cone of which this frustum is a part.

8. Describe the solid generated by a right triangle if it is revolved about its hypotenuse.

9. Show how a frustum of a cone of revolution can be generated by revolving a certain type of quadrilateral about one of its sides.

10. A plane section of a cone parallel to the base bisects the altitude. If the area of the base is 8 sq. in., what is the area of this plane section?

11. The altitude of a cone of revolution is 24 in. and the radius of the base is 6 in. How far from the base must a plane parallel to the base be drawn in order to cut the cone and form a section whose area is $16\pi$ sq. in.?

12. The altitude of a cone is $h$ and the area of the base is $b$. How far from the vertex must a plane parallel to the base be drawn in order to cut the solid and form a section whose area is $\frac{b}{64}$? $\frac{b}{9}$? $\frac{b}{n}$?

13. A line $HB$ is perpendicular to side $AB$ of a $\triangle ABC$ and is coplanar with the triangle. Describe the solid generated by $\triangle ABC$ if it is revolved about $HB$ (Fig. 166).

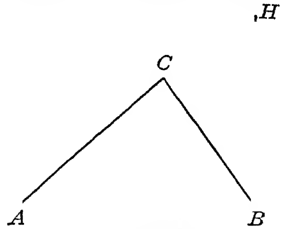


Fig. 166

14. $CD$ is perpendicular to side $AB$ extended of a $\triangle ABC$. Describe the solid generated by $\triangle ABC$ if it is revolved about $CD$ (Fig. 167).

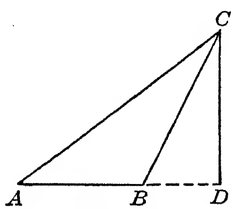


Fig. 167


15. $HB$ is perpendicular to side $AB$ of a parallelogram $ABCD$ and is coplanar with $ABCD$. Describe the solid which is generated by $ABCD$ if it is revolved about $HB$ (Fig. 168).

FIG. 168

**146. The Conic Sections.** Choose any right circular cone and extend the elements through the vertex so as to form a piece of the upper nappe of the conic surface.

Draw a plane $S$ parallel to $M$, the plane of the base, so as to cut the conic surface. Then by § 143, the section thus formed will be a *circle* (Fig. 169).

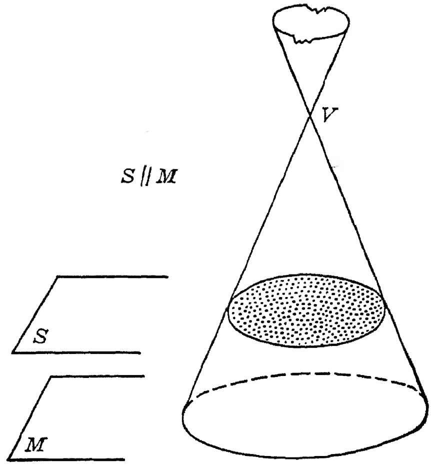



FIG. 169

Draw a plane $T$ parallel to an element and cutting the conic surface. The curve which is the intersection of $T$ with the conic surface is called a *parabola* (Fig. 170).

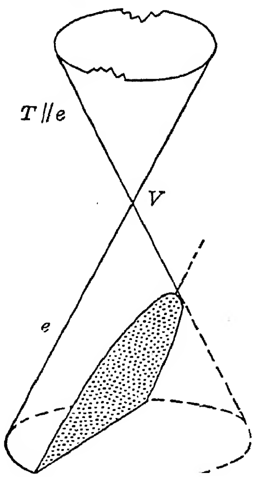



FIG. 170

Draw a plane $U$ cutting the conic surface, and choose $U$ in such a way that the inclination of $U$ to $M$ is an angle *less* than the inclination of any element to $M$. The curve of intersection of $U$ with the conic surface is called an *ellipse* (Fig. 171).

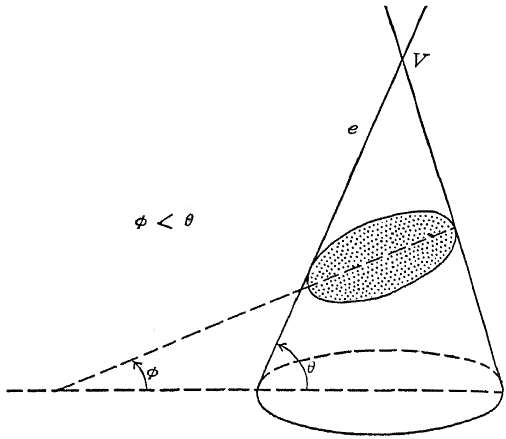



Fig. 171

Finally, draw a plane $W$ cutting the conic surface, and choose $W$ in such a way that the inclination of $W$ to $M$ is an angle *greater* than the inclination of any element to $M$. Plane $W$ will then cut *both* nappes of the conic surface. The two curves of intersection of plane $W$ with the surfaces are the two branches of an *hyperbola* (Fig. 172).

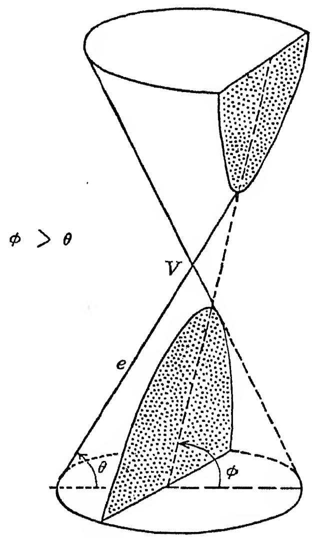



Fig. 172

In Figs. 169–172 if the cutting plane happens to contain $V$ the resulting intersections will not be the curves just described, but instead will be a single point, a single straight line, or two intersecting straight lines. Discuss each of these abnormal or "degenerate" cases.

A mathematical treatment of these four *conic sections:* the circle, the parabola, the ellipse and the hyperbola is beyond the scope of elementary geometry.

A systematic study of them together with their properties and measurement belongs to a higher branch of mathematics known as Analytical Geometry.

**147. The Generalized Conic Solid** (Fig. 173). The concept of a generalized conic surface is similar to that of the generalized cylindric surface discussed in § 123.

Let $f$ be any plane figure: polygon, curve, or a figure bounded by any combination of straight line segments and arcs; let $V$ be a fixed point not coplanar with $f$; let $e$ be a straight line containing $V$ and touching the boundary of $f$. As $e$ is allowed to move in accordance with this restriction it will generate a surface similar to the conic surface already mentioned.

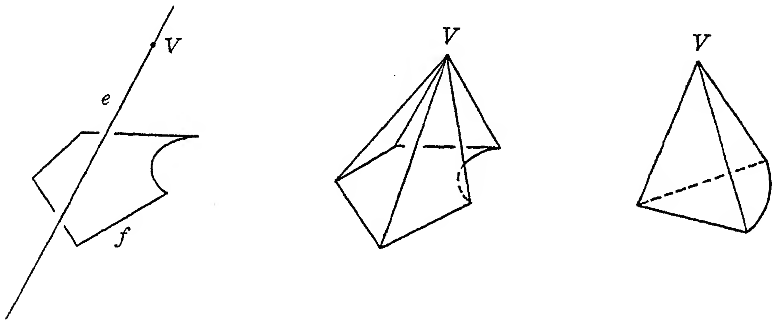



FIG. 173

If a solid is formed by passing a plane through all the elements of this surface, the solid bounded by this plane and one nappe of the surface may be thought of as a *generalized conic solid*. Later we shall assume that the volume formulas to be derived for pyramids and cones are valid for these generalized conic solids.

*Chapter Ten*

# PYRAMIDS AND CONES: AREAS AND VOLUMES

**148.** THEOREM 39.

The lateral area of a regular pyramid is one-half the product of the perimeter of the base by the slant height. $S = \frac{1}{2}pf.$

(Area of each lateral face $\frac{1}{2}fx$. Add the areas of the lateral faces.)

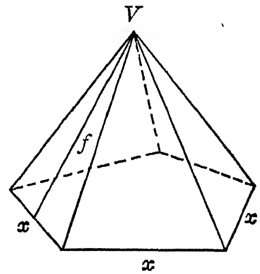



FIG. 174

**149.** THEOREM 40.

The lateral area of a frustum of a regular pyramid is the product of one-half the sum of the perimeters of the bases by the slant height.

$$S = \frac{1}{2}(p_1 + p_2)f.$$

(Add together the areas of the congruent trapezoids which make up the lateral surface.)

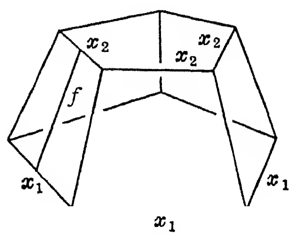



FIG. 175

**150. Tangent Plane.** A plane is *tangent* to a conic surface if the two surfaces have one and only one element of the conic surface in common. (Compare with § 102.)

**151. Circumscribed and Inscribed Pyramids.** If the base of a pyramid is coplanar with the base of a cone, and if the lateral faces of the pyramid are each tangent to the wall of the cone, the pyramid is said to be *circumscribed* about the cone, — or the cone is *inscribed* in the pyramid (Fig. 176).

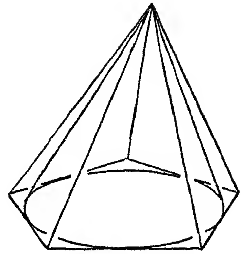


FIG. 176

If the base of a pyramid is coplanar with the base of a cone, and if the lateral edges of the pyramid are elements of the cone, the pyramid is said to be *inscribed* in the cone, — or the cone is *circumscribed* about the pyramid (Fig. 177).

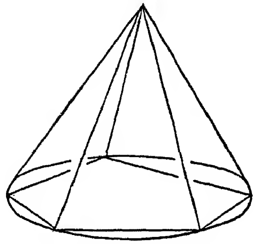


FIG. 177

**152. Circumscribed and Inscribed Frustums of Pyramids.** Study the figures and deduce the obvious definitions. Fig. 178 shows a frustum of a

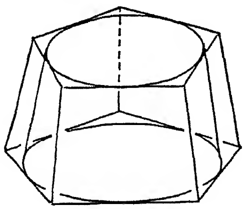


FIG. 178

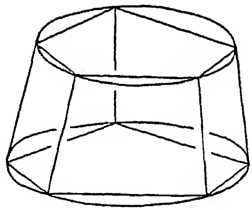


FIG. 179

pyramid *circumscribed* about a frustum of a cone. Fig. 179 shows a frustum of a pyramid *inscribed* in a frustum of a cone.

**153.** The following facts which relate to the foregoing are easily established:

A. If a pyramid is circumscribed about or inscribed in a cone, the vertices of the two solids coincide.

B. A regular pyramid may be circumscribed about or a regular pyramid may be inscribed in a given right circular cone. Conversely, a right circular cone may be circumscribed about or a right circular cone may be inscribed in a given regular pyramid.

C. If a regular pyramid is circumscribed about a right circular cone, the lines of tangency of the two lateral surfaces are slant heights of the pyramid.

D. A frustum of a regular pyramid may be circumscribed about or inscribed in a frustum of a right circular cone. Conversely, a frustum of a right circular cone may be circumscribed about or inscribed in a frustum of a regular pyramid.

E. If a frustum of a regular pyramid is circumscribed about a frustum of a right circular cone, the lines of tangency of the two lateral surfaces are slant heights of the frustum of the pyramid.

**154.** Assumption.

A. If a pyramid is circumscribed about or inscribed in a cone, and if the number of lateral faces of the pyramid is caused to become infinite,
   (i) the lateral area and the total area of the pyramid approach as limits, respectively, the lateral area and total area of the cone;
   (ii) the volume of the pyramid approaches as a limit the volume of the cone.

B. If a frustum of a pyramid is circumscribed about or inscribed in a frustum of a cone, and if the number of lateral faces of the frustum of the pyramid is caused to become infinite,
   (i) the lateral area and total area of the frustum of the pyramid approach as limits, respectively, the lateral area and total area of the frustum of the cone;
   (ii) the volume of the frustum of the pyramid approaches as a limit the volume of the frustum of the cone.

Compare the above with § 108. Just as a cylinder may be regarded as the limiting form of a prism, so a cone may be regarded as the limiting form of a pyramid.

**155.** Theorem 41.

The lateral area of a cone of revolution is one-half the product of the circumference of the base by the slant height (element). $S = \frac{1}{2}ce$.

(Circumscribe a regular pyramid about the cone. Use § 154 and Ref. 91, and proceed in general as in § 110.)

**156. Corollary A** (Th. 41.)

In a cone of revolution of radius $r$ and element $e$:

$$\text{lateral area, } S = \pi re;$$
$$\text{total area, } T = \pi r^2 + \pi re.$$

**157. Corollary B** (Th. 41).

The lateral areas or the total areas of two similar cones of revolution are to each other as the squares of their respective elements, radii, altitudes.

(Compare with § 112.)

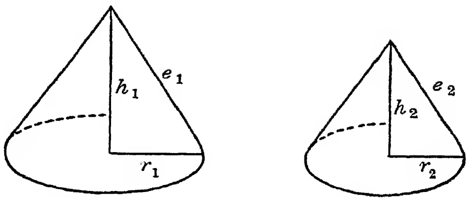



FIG. 180

**158.** THEOREM 42

The lateral area of a frustum of a cone of revolution is the product of one-half the sum of the circumferences of the bases by the slant height (element). $S = \frac{1}{2}(c_1 + c_2)e.$

(Circumscribe a frustum of a regular pyramid about the given frustum. Use § 154 and Ref. 91, and proceed as in § 155.)

**159. Mid-Section of a Frustum.** If a plane parallel to the bases of a frustum (either of a pyramid or of a cone) is drawn to cut all the elements of the frustum, the section thus formed is called a *mid-section* of the frustum.

## EXERCISES

*Group Fourteen*

1. Prove that the lateral area of a frustum of a pyramid or cone is the product of the perimeter of its mid-section by the slant height of the frustum.

2. Find the lateral area of a cone of revolution with radius 3 in. and altitude 4 in.

3. $P$ and $Q$ are any two pyramids having equal altitudes and bases of equal area. In each take a plane section parallel to the base; in both cases let this section be taken at the same distance from the respective bases. What is true of the areas of these plane sections?

4. Find the lateral area of a regular hexagonal pyramid each basal edge of which is 10 in., and each lateral edge of which is 13 in.

5. In Ex. 4 find the lateral areas of the inscribed and circumscribed cones.

6. The sides of a triangle are 5 in., 12 in., 13 in. Find the lateral area of the cone of revolution generated by revolving this triangle through 360° about (a) its 12-in. side; (b) its 5-in. side.

7. The slant height of a regular square pyramid is 24 in. and the lateral area is 960 sq. in. Find the area of the base.

8. The bases of a frustum of a regular square pyramid are 6 in. and 8 in. on a side, respectively. The slant height is 5 in. Find the lateral area.

**9.** The altitude of a regular square pyramid is 30 in. and the area of the base is 1024 sq. in. A plane is drawn parallel to the base and bisects the altitude. Find the lateral area of the frustum thus formed.

**10.** The slant height of a frustum of a regular triangular pyramid is 10 in. and the areas of the bases are respectively $4\sqrt{3}$ in. and $9\sqrt{3}$ in. Find the lateral area of the frustum.

**11.** The radii of the bases of a frustum of a cone of revolution are respectively 2 in. and 3 in., and the slant height is 4 in. Find the lateral area of the frustum.

**12.** The diameters of the bases of a frustum of a cone of revolution are respectively 10 in. and 16 in., and the altitude is 4 i . Find the lateral area. What is the area of any axial section?

**13.** The slant height of a frustum of a regular triangular pyramid is 6 in., and the lateral area is 126 sq. in. Find the area of the mid-section of the frustum.

**14.** The lateral areas of two similar cones of revolution are respectively 9 sq. in. and 16 sq. in. If the radius of the smaller is 5 in., what is the radius of the larger?

**15.** The altitude of one of two similar cones of revolution is $k$ times the altitude of the second. What is the ratio of the lateral area of the larger cone to that of the smaller?

**16.** The sum of the lateral areas of two similar cones of revolution is 52 sq. in. The ratio of their respective elements is $\frac{2}{3}$. Find the lateral area of each cone.

**17.** A regular pyramid (or right circular cone) has an altitude $h$ and a base $b$. A plane parallel to the base bisects the altitude. Prove that the lateral area of the solid is the product of the slant height by the perimeter of the parallel section. (This section is a *mid-section* of the pyramid or cone.)

**18.** The area of the base of a cone of revolution is $36\pi$ sq. in. The altitude is 8 in. Find the lateral areas of the inscribed and circumscribed regular triangular pyramids.

**19.** The sides of a triangle are 10 in., 17 in., 21 in. The triangle is revolved through 360° about its longest side. Find the total area of the solid generated by this triangle.

**20.** Find the radius of a cone of revolution of slant height 6 in. in which the total area is $1\frac{1}{2}$ times the lateral area.

**160.** Let $T\text{–}ABC$ be any pyramid. Let $h$ = altitude, $b$ = area of base. Divide $h$ into any number of equal segments. Through each point of division draw a

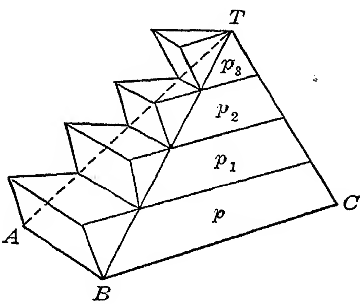



FIG. 181

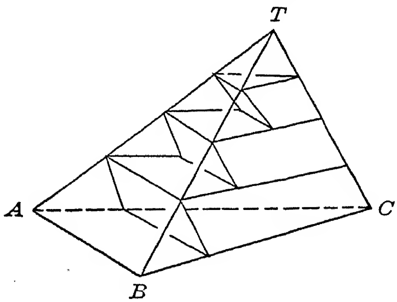



FIG. 182

plane parallel to the base. Using $b$ and each of the parallel sections as bases construct prisms as shown in Fig. 181 and Fig. 182.

In Fig. 181, starting with the first, name the prisms $p$, $p_1$, $p_2$, $p_3$.

Show that the prisms of Fig. 182 are respectively equal to the prisms $p_1$, $p_2$, $p_3$ (§§ 131, 119).

Represent volumes as follows:

Let $p + p_1 + p_2 + p_3 = W$.
Let $T\text{–}ABC = V$.
Let $p_1 + p_2 + p_3 = U$.
Then $W > V$ and $V > U$ (Ax. 3).
Also $W - U = p$.

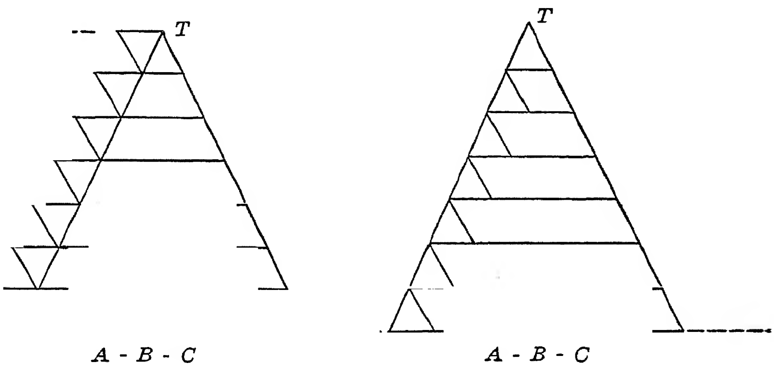



FIG. 183

Now increase the number of divisions of $h$ indefinitely. $W$ now represents the sum of $p$ (which has become smaller) and the remaining prisms of Fig. 181. $U$ represents the sum of all prisms except $p$. (See Fig. 183.) $p$ approaches 0 in value.

$$\text{But} \quad (W - V) < p \quad \text{and} \quad (V - U) < p \quad \text{at all times.}$$

Therefore, since $p$ approaches 0, $W$ must approach $V$ in value, and $U$ also must approach $V$ in value.

*That is, the sum of the volumes of the prisms either of Fig. 181 or of Fig. 182 approaches as a limit the volume of the pyramid $T\text{–}ABC$.*

**161.** Let $T\text{–}ABC$ and $O\text{–}XYZ$ be any two pyramids having equal altitudes ($h$), and bases of equal area ($b$). Let $V$ = volume of $T\text{–}ABC$; let $W$ = volume of $O\text{–}XYZ$ (Fig. 184). We shall prove: $V = W$.

Divide the altitude of each pyramid into any number of equal parts. Through each point of division draw a plane parallel to the base, and construct prisms as in Fig. 181.

Let the sum of the volumes of the prisms of $T\text{–}ABC$ be $V'$; let the sum of the volumes of the prisms of $O\text{–}XYZ$ be $W'$.

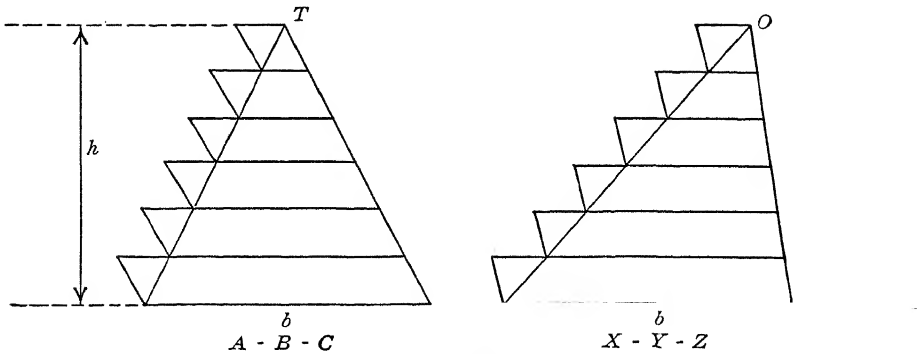



FIG. 184

Now show that $V' = W'$ (§§ 131, 119).

Let the number of divisions of each altitude increase; that is, let the number of prisms of the two pyramids increase indefinitely and at the same rate.

By § 160: $$V' \longrightarrow V$$
and $$W' \longrightarrow W.$$

But the two variables $V'$ and $W'$ are always equal.

Therefore, their limits $V$ and $W$ must be equal (Ref. 91).

Therefore we have:

*Two pyramids must have equal volumes if their altitudes are equal and if their bases are equal in area.*

**162.** THEOREM 43.

The volume of any pyramid is one-third the product of the area of its base by its altitude. $V = \frac{1}{3}bh.$

*Given:* Any pyramid.
$b$ = base, $h$ = altitude
$V$ = volume.

*Prove:* $V = \frac{1}{3}bh.$

Part I. Let $E$–$ABC$ be a triangular pyramid (Fig. 185).

Using $ABC$ as base and $BE$ as lateral edge construct a prism $ABC$–$DEF$. Pass planes through the sets of points: $E$, $A$, $C$ and $E$, $A$, $F$, thus dividing the prism into the three pyramids: $E$–$ABC$, $E$–$ADF$, $E$–$ACF$.

Use §161 to show:

1) $E$–$ADF$ = $E$–$ACF$
2) $E$–$ABC$ = $E$–$ADF$

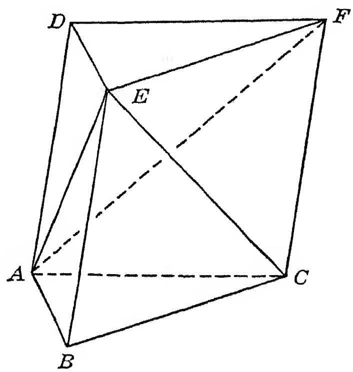



FIG. 185

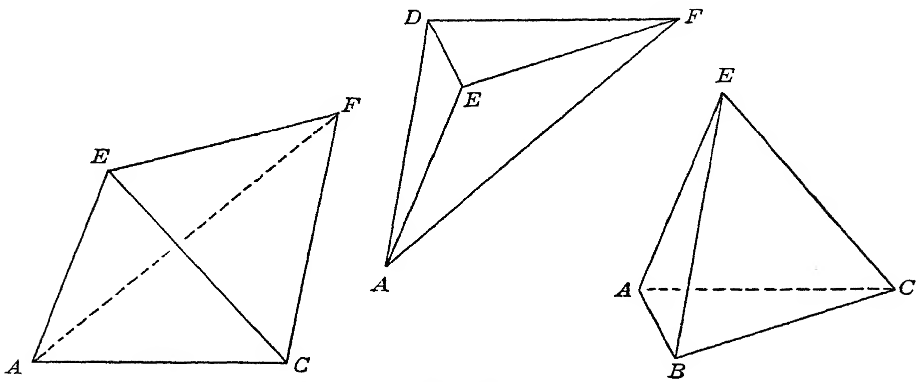



FIG. 186

(Study figures 185, 186 and note that $E$–$ADF$ is actually the same pyramid as $A$–$DEF$.)

$\therefore$ the three pyramids are equal, and hence each one must be one-third of the entire prism.

$\therefore$ $E\text{–}ABC = \frac{1}{3}ABC\text{–}DEF = \frac{1}{3}(bh) = \frac{1}{3}bh$ (§ 119).

### PART II

If the given pyramid is not triangular, divide it into triangular pyramids by planes through the vertex and diagonals of the base (Fig. 187).

Obtain the volumes of these triangular pyramids and add the results.

*Note:* An alternative proof of Theorem 43 follows. This second proof is more algebraic in nature, being an interesting application of infinite series to geometry.

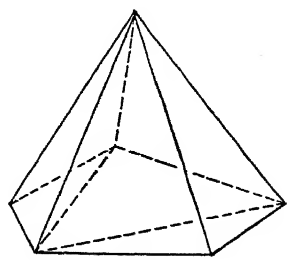


FIG. 187

**162.** THEOREM 43.

The volume of a pyramid is one-third the product of the area of its base by its altitude. $V = \frac{1}{3}bh$.

*Given:* Any pyramid.
$b$ = area of base
$h$ = altitude
$V$ = volume

*Prove:* $V = \frac{1}{3}bh$.

1) Divide $h$ into $n$ equal segments, each equal to $x$. Through each point of division pass a plane parallel to the base of the pyramid, thus forming a

series of parallel sections. Starting with the section nearest to the base call the areas of these sections $b_2$, $b_3$, $b_4$, . . . Then the area of the topmost section will be $b_n$.

2) Upon $b$, $b_2$, $b_3$, $b_4$, . . . as bases construct prisms of the type discussed in § 160. Let $P$ be the sum of the volumes of these prisms.

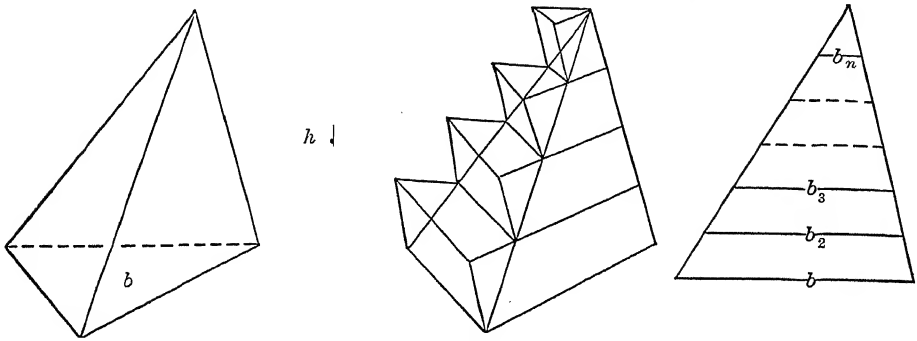



FIG. 188

3) $$P = bx + b_2x + b_3x + b_4x + \cdot\cdot\cdot + b_nx$$

$$= bx + \left(\frac{n-1}{n}\right)^2 bx + \left(\frac{n-2}{n}\right)^2 bx + \left(\frac{n-3}{n}\right)^2 bx + \cdots + \left(\frac{1}{n}\right)^2 bx \quad (\S\S\ 131,\ 119)$$

$$= bx\left[\left(\frac{n}{n}\right)^2 + \left(\frac{n-1}{n}\right)^2 + \left(\frac{n-2}{n}\right)^2 + \left(\frac{n-3}{n}\right)^2 + \cdot\cdot\cdot + \left(\frac{1}{n}\right)^2\right]$$

$$= \frac{bx}{n^2}\left[n^2 + (n-1)^2 + (n-2)^2 + (n-3)^2 + \qquad + 1^2\right].$$

In the brackets each term which is squared is seen to be 1 less than the term before it which is squared. The largest of these is $n$, and the smallest is 1. Hence the series in the brackets is actually the series: $1^2 + 2^2 + 3^2 + 4^2 + \cdot\cdot\cdot + n^2$. From Ref. 92 the formula for the sum of the $n$ terms of this series is

$$\frac{n}{6}(2n+1)(n+1).$$

4) ∴ $$= \frac{bx}{n^2}\left[\frac{n}{6}(2n+1)(n+1)\right] = \frac{bxn}{n^2}\left[\frac{(2n+1)(n+1)}{6}\right]$$

$$= \frac{b(nx)}{3}\left[\frac{(2n^2+3n+1)}{2n^2}\right] = \frac{bh}{3}\left[1 + \frac{3}{2n} + \frac{1}{2n^2}\right].$$

5) Now let $n$ become infinite. As this occurs, the fractions $\frac{3}{2n}$ and $\frac{1}{2n^2}$ each approach 0 in value. At the same time $P$ approaches $V$ as a limit.

6) ∴ $V = \frac{bh}{3}$, by Ref. 91.

**163. Corollary A** (Th. 43).

The volume of a cone is one-third the product of the area of its base by its altitude. $V = \frac{1}{3}bh.$

(Circumscribe a pyramid about the given cone, or else inscribe a pyramid in the cone. Apply § 154 A(ii) and Ref. 91.)

**164.** THEOREM 44.

If the areas of the bases of a frustum of a pyramid are $b_1$ and $b_2$, and if the altitude is $h$, the volume is given by the formula:

$$V = \frac{h}{3}(b_1 + b_2 + \sqrt{b_1 b_2}).$$

*Given:* Frustum of any pyramid.
$b_1$ and $b_2$, areas of bases
$h$ = altitude
$V$ = volume

*Prove:* $V = \frac{h}{3}(b_1 + b_2 + \sqrt{b_1 b_2}).$

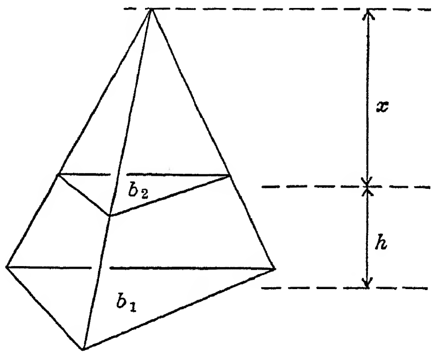


FIG. 189

1) Extend the lateral edges of the frustum so as to form the pyramid of which the given frustum is a part. Let $(x + h)$ be the altitude of this pyramid (Fig. 189).
2) Then $V$ = Vol. entire pyramid − Vol. small pyramid
$= \frac{1}{3}(x + h)b_1 - \frac{1}{3}b_2 x$ (§ 162)
$= \frac{1}{3}[hb_1 + (b_1 - b_2)x].$
3) But $\dfrac{b_2}{b_1} = \dfrac{x^2}{(x+h)^2}$ or $\dfrac{x}{(x+h)} = \dfrac{\sqrt{b_2}}{\sqrt{b_1}}$ (§ 131).
4) In step 3, solve for $x$. Show that $x = \dfrac{h(\sqrt{b_1 b_2} + b_2)}{b_1 - b_2}.$
5) Substitute this value of $x$ in the last line of step 2, and obtain: $V = \frac{1}{3}(hb_1 + h[\sqrt{b_1 b_2} + b_2])$
or: $V = \frac{h}{3}(b_1 + b_2 + \sqrt{b_1 b_2}).$

**165. Corollary A** (Th. 44).

If the areas of the bases of a frustum of a cone are $b_1$ and $b_2$, and if the altitude is $h$, the volume is: $V = \frac{h}{3}(b_1 + b_2 + \sqrt{b_1 b_2}).$

(Circumscribe a frustum of a pyramid about the frustum of the cone, or else inscribe a frustum of a pyramid in the given frustum of a cone. Apply § 154 B(ii) and Ref. 91.)

## EXERCISES

### *Group Fifteen*

1. Find the volume of a square pyramid if the altitude is 20 in. and each basal edge is 12 in.

2. Find the volume of a cone if the altitude is 14 in. and the base is a circle of radius 6 in.

3. If two pyramids (or cones) have equal bases, their volumes are to each other as their altitudes. If two pyramids (or cones) have equal altitudes, their volumes are to each other as the areas of their bases. Prove.

4. A cone not fixed in shape but having a constant volume has a base $b$ which is fixed in position and area. What is the locus of the vertex of the cone?

5. A cone and a cylinder have equal bases and equal volumes. How do their altitudes compare?

6. The base of a cone is two-thirds the base of a cylinder, and their volumes are equal. Compare their altitudes.

7. Find the volume of a regular hexagonal pyramid each basal edge of which is 6 in. and the altitude of which is 10 in.

8. Find the volume of a cone of revolution if the altitude is 12 in. and the slant height (element) is 13 in.

9. The altitude of a regular triangular pyramid is 5 in. and the volume is $15\sqrt{3}$ cu. in. Find the length of each basal edge.

10. Find the volume of a frustum of a square pyramid if the altitude is 9 cm., and if the bases are squares whose edges are respectively 8 cm. and 6 cm.

11. If the radii of the bases of a frustum of a circular cone are $r_1$ and $r_2$, and if the altitude is $h$, show that the volume is

$$\tfrac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2).$$

12. The radii of the bases of a frustum of a cone of revolution are respectively 7 in. and 10 in. and the element of the frustum is 5 in. Find the volume.

13. A $\triangle ABC$, right-angled at $C$, is revolved through 360° about the leg $BC$ as an axis. $BC = 8$ in., $CA = 6$ in. Find the lateral area, total area, and volume of the solid thus generated.

14. In Ex. 13 find the volumes of the regular triangular pyramids which can be respectively inscribed in and circumscribed about the cone.

15. $ABCD$–$EFGH$ is a frustum of a regular square pyramid. $AB = 26$, $EF = 10$, the altitude $= 15$. Find the lateral area and the volume.

16. The bases of a frustum of a right circular cone are circles whose diameters are respectively 18 in. and 4 in. Find the lateral area and volume if the slant height is 25 in.

17. Find the volume of the solid mentioned in Ex. 19, Group 14.

18. Find the volume of a regular octagonal pyramid each basal edge of which is 4 ft. and the altitude of which is 8 ft.

19. The volume of a regular triangular pyramid is $300\sqrt{3}$ and the area of its base is $75\sqrt{3}$. Find the lateral area.

20. The volume of one of two similar cones of revolution is 64 times that of the other. Find the ratio of the radii (larger to smaller), and the ratio of their lateral areas (larger to smaller).

21. The lateral area of one of two similar cones of revolution is $\frac{1}{9}$ that of the other. Find the ratio of the volume of the first to that of the second.

22. The radius of one of two similar cones of revolution is $\frac{2}{3}$ that of the other. The sum of their volumes is 140 cu. cm. Find the volume of each.

23. The altitude of a pyramid is 12 cm. How far from the vertex must a plane parallel to the base be drawn in order to separate the pyramid into two solids of equal volume?

24. The altitude of a cone is $h$ and its volume is 162 cu. cm. A plane $S$ parallel to the base cuts the altitude at a distance $h/3$ above the base. Find the volume of the frustum included between $S$ and the base.

25. The altitude of a pyramid $P$ is $3x$. The pyramid is separated into three solids $A$, $B$, $C$ and by two planes which are parallel to the base and which trisect the altitude. Find the ratio of the volume of $A$ to that of $P$; $B$ to $P$; $C$ to $P$.

26. The edge of a regular tetrahedron is $e$. Show that its altitude is $\frac{e}{3}\sqrt{6}$, that its total area is $e^2\sqrt{3}$, and that its volume is $\frac{e^3}{12}\sqrt{2}$.

27. Find the volume of a regular tetrahedron whose edge is 6 in.

28. The total area of one regular tetrahedron is $289\sqrt{3}$ and the volume of a second is $144\sqrt{2}$. Find the edge and altitude of each.

29. A container is built in the form of a right circular cone. Any axial section is an equilateral triangle. (Such cones are often called *equilateral cones.*) Find to the nearest hundredth of an inch the lengths of the radius and altitude, respectively, if the cone is to contain 1 gallon (231 cu. in.).

30. A circular sector has a radius of 20 in. and an angle of 120°. If this sector is cut out of paper and rolled so as to form the lateral surface of a right circular cone, find the total area and volume of the cone. Give your answer correct to the nearest tenth.

31. In a trapezoid $ABCD$, $AB = 9$ ft., $\angle B = 90°$, $BC = 4$ ft., $CD = 6$ ft. $ABCD$ is revolved through 360° about $BC$ as an axis. Find the lateral area and volume of the solid thus generated.

32. In $\triangle ABC$, $AB = 11$, $BC = 20$, $CA = 13$. $h$ is a fixed line perpendicular to $AB$ at $B$. $\triangle ABC$ is always coplanar with $h$, and is revolved through 360° about $h$ as an axis. Find the volume of the solid generated by $\triangle ABC$ (Fig. 190).

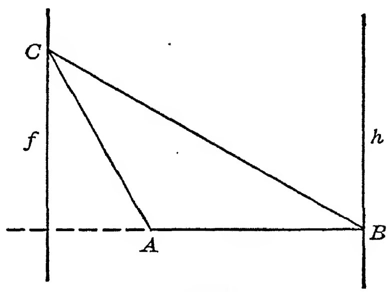



FIG. 190

33. In Ex. 32 find the volume of the solid generated by $\triangle ABC$ if it is revolved about a fixed line $f$ which is coplanar with $\triangle ABC$, which contains point $C$, and which is perpendicular to $BA$ extended.

34. In a parallelogram $ABCD$, $AB = 20$ in., $AD = 12$ in., $\angle A = 60°$. A fixed line $h$ is perpendicular to $AB$ at $A$. $ABCD$ remains coplanar with $h$, and is revolved through 360° about $h$ as an axis. Find the total exposed area of the solid generated by $ABCD$ (Fig. 191).

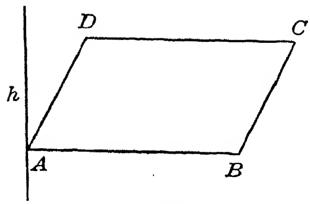



FIG. 191

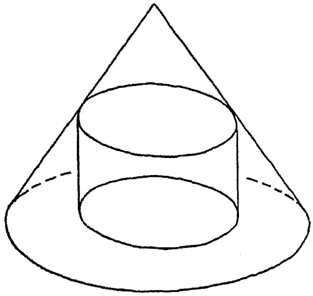


FIG. 192

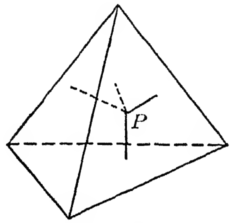



FIG. 193

35. A block of wood is in the form of a right circular cone. The altitude is 8 in. and the radius of the base is 6 in. A cylindrical hole 6 in. in diameter is bored completely through the solid, the axis of the hole coinciding with the axis of the cone. Find the amount of wood left after the hole is bored (Fig. 192).

36. From any point $P$ within a regular tetrahedron lines $x$, $y$, $z$, $w$ are drawn perpendicular to the four faces, respectively. If $h$ is the altitude of the tetrahedron, prove that $x + y + z + w = h$. (From $P$ draw lines to the four vertices thus forming four pyramids. What is the volume of each pyramid? To what is the sum of these volumes equivalent?)

37. Parallelogram $ABCD$ is the base of a pyramid $V\text{–}ABCD$. $AE$ bisects $BC$ and cuts $DB$ at $F$. What is the ratio of the volume of pyramid $V\text{–}FBE$ to that of pyramid $V\text{–}ABCD$ (Fig. 194)?

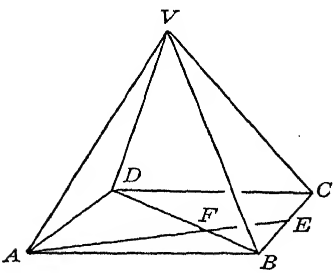



FIG. 194

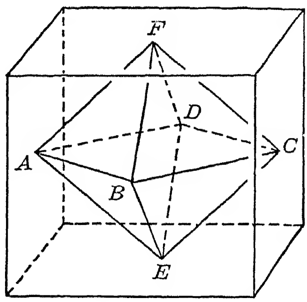



FIG. 195

38. Points $F$, $A$, $B$, $C$, $D$, $E$ are respectively the centers of the faces of a cube whose edge is $10\sqrt{2}$. Find the volume of the solid whose vertices are $F, A, B, C, D, E$ (Fig. 195).

**39.** A chimney pot is built in the form of a frustum of a regular square pyramid on the outside, and the flue is a hole in the form of a frustum of a right circular cone. $AB = 3$ ft., $EF = 2$ ft., the upper diameter of the hole is 1 ft., the lower diameter is 2 ft. The altitude of the figure is 4 ft. Find the amount of material used in constructing the chimney pot (Fig. 196).

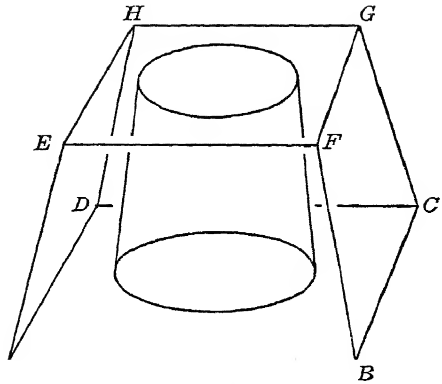



FIG. 196

**40.** $V\text{–}ABCD$ is a regular square pyramid. $AB = 12$ in. The altitude is $6\sqrt{3}$ in. $VX = \frac{1}{3}VA$. Through $X$ a plane $S$, parallel to $AD$ and perpendicular to the plane of $VAD$ is drawn, forming the section $XYZW$. Find the volume of the pyramid $V\text{–}XYZW$ and the volume of the solid $ABCD\text{–}XYZW$ (Fig. 197).

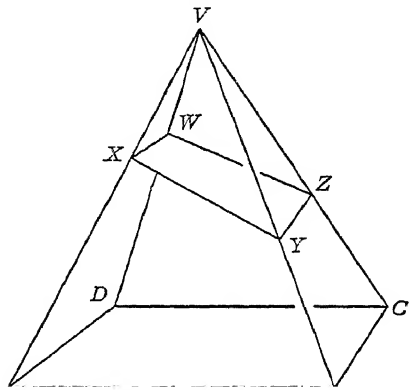
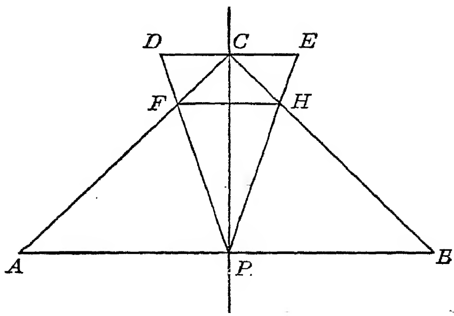



FIG. 197

**41.** In Ex. 40 find the lateral area of the pyramid $V\text{–}XYZW$.

**42.** In $\triangle ABC$, $CA = CB$, $P$ is the midpoint of $AB$. $AB = 40$ in. $DCE$ is parallel to $AB$. $DC = CE = 5$ in. The entire figure is revolved through 180° about $CP$ as an axis. Find the area of the surface generated by the line $FH$ (Fig. 198).



# *Chapter Eleven*

## POLYHEDRAL ANGLES. POLYHEDRONS

**166. Polyhedral Angles.** When three or more planes meet in a common point $V$ the figure thus formed is called a *polyhedral angle*. (See Fig. 199.)

The *vertex* is the common point $V$. The *edges* are the intersections of the planes already mentioned, and are seen to be lines which emanate from $V$ ($VA$, $VB$, $VC$, etc.). The *faces* are the portions of the given planes which are included between consecutive edges. (Note that the vertex, faces, and edges are respectively the vertex, lateral faces, and lateral edges of a pyramidal surface of one nappe.) The *face angles* are the angles having $V$ as a common vertex and consecutive edges as sides. ($\angle AVB$, $\angle BVC$, $\angle CVD$, etc., are face angles.) Inherent in the concept of polyhedral angle just presented is the fact that *each face angle is necessarily less than 180°.*

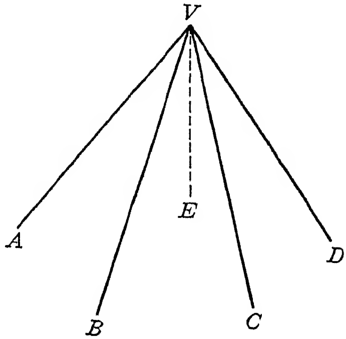



FIG. 199

A polyhedral angle is *convex* if it is formed by a pyramidal surface whose directrix is any ordinary convex polygon; otherwise it is concave. Only the *convex* type will be considered here.

| | | | | |
|---|---|---|---|---|
| | trihedral | | three | |
| A polyhedral angle is | tetrahedral | if it has | four | faces, etc. |
| | pentahedral | | five | |
| | hexahedral | | six | |

Note that if a polyhedral angle possesses $n$ faces, it must also possess $n$ edges and hence $n$ dihedral angles. The edges and faces of these dihedral angles are edges and faces of the given polyhedral angle. (See dh $\angle A\text{–}VB\text{–}C$, dh $\angle B\text{–}VC\text{–}D$, etc., in Fig. 199.)

**167.** THEOREM 45.

The sum of any two face angles of a trihedral angle is greater than the third face angle.

$V\text{–}XYZ$ is the given trihedral angle. $XVY$, $YVZ$, $ZVX$ are the face angles.

(a) In case all the face angles are equal the theorem is obvious.

(b) In case two of the face angles are equal and each is greater than the third, the theorem is again obvious.

(c) Suppose that the face angles are all unequal. Suppose that $XVZ$ is the largest. It is then obvious that $XVZ + YVZ > XVY$ and that $XVZ + XVY > YVZ$. Hence we need to prove: $XVY+YVZ > XVZ$.

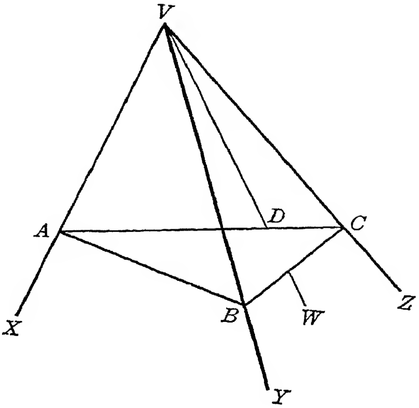



FIG. 200

1) In face $XVZ$ draw line $VW$ making $\angle XVW = \angle XVY$.
2) On $VY$ and $VW$, respectively, take $VB = VD$. Draw any plane $S$ containing points $B$ and $D$ and cutting $VX$ at $A$ and $VZ$ at $C$.
3) Obtain $\triangle AVB \cong \triangle AVD$.
4) $\therefore\ AD = AB$.
5) $AB + BC > AC$ (Post. 2).
6) $\therefore\ BC > DC$ (Ax. 7).
7) $\therefore\ \angle BVC > \angle DVC$ (Ref. 18).
8) $\therefore\ \angle AVB + \angle BVC > \angle AVD + \angle DVC$.
9) $\therefore\ \angle XVY + \angle YVZ > \angle XVZ$ (Ax. 3).

(d) In case any two face angles are equal and each is less than the third, the proof is similar to that for (c).

**168.** THEOREM 46.

The sum of the face angles of any polyhedral angle is less than 360°

*Given:* Ph $\angle V$, having $n$ faces.
F = sum of face angles.

*Prove:* $F < 360°$.

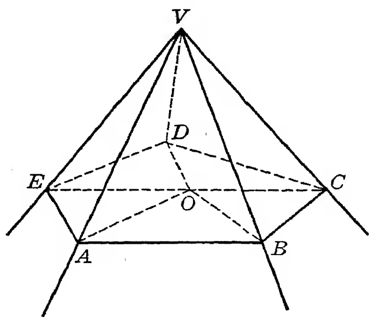



FIG. 201

1) Draw a plane $M$ cutting all the edges of $V$, thus forming a pyramid $V\text{–}ABCD$ . . .
2) In plane $M$ and within the base of the pyramid choose any point $O$ and draw $OA$, $OB$, $OC$, etc. The base is now divided into $n$ triangles.
3) The sum of the angles around $O$ is 360°.
4) Represent angle sums (degrees) as follows:

$T$ = sum of all the angles $VAB$, $VBA$, $VBC$, $VCB$, $VCD$, etc.;
$S$ = sum of all the angles $OAB$, $OBA$, $OBC$, $OCB$, $OCD$, etc.

5) Then in polygon $ABCD\ldots$: $S + 360° = n(180°)$ (Ref. 29 and Ax. 4).
In the lateral faces: $F + T = n(180°)$

$$\therefore\ F + T = S + 360° \quad \text{(Ax. 1)}.$$

6) or: $F = 360° - (T - S)$.

7) But at vertex $B$, for example, $\angle VBA + \angle VBC > \angle OBA + \angle OBC$ (§167); similarly, for vertices $C$, $D$, $E$, etc.

8) Adding: $T > S$ (Ax. 5).

9) $\therefore$ $T - S$ must be greater than zero.

10) Return to step 6. $F$ is now seen to equal 360° diminished by a positive amount $(T - S)$.

That is, $F < 360°$

**169.** Theorem 47.

> In any trihedral angle:
> (a) each dihedral angle is less than 180°;
> (b) the sum of the dihedral angles is less than 540°;
> (c) the sum of the dihedral angles is greater than 180°.

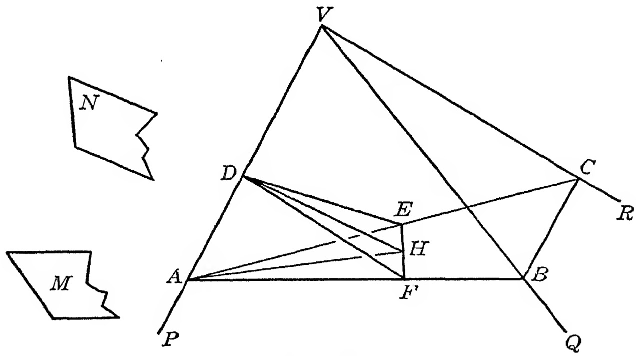



Fig. 202

*Given:* trh $\angle V\text{–}PQR$
dh $\angle VP = x°$
dh $\angle VQ = y°$
dh $\angle VR = z°$

*Prove:* (a) $x < 180°$, $y < 180°$, $z < 180°$;
(b) $x + y + z < 540°$;
(c) $x + y + z > 180°$.

1) On $VP$, $VQ$, $VR$ respectively, choose $A$, $B$, $C$ so that $VA = VB = VC$. Through $A$, $B$, $C$ draw a plane $M$. Then plane $M$ must be oblique to all three lines $VA$, $VB$, $VC$ (Fig. 202).

2) Through any point $D$ on $VA$ draw a plane $N$ perpendicular to $VA$, cutting face $VAB$ in $DF$, face $VAC$ in $DE$, and $M$ in $EF$. Then $\angle EDA = \angle FDA = 90°$. Hence $\angle EDF$ is the plane angle of dh $\angle VP$. $\therefore\ \angle EDF = x°$.

(a)

3) $D\text{–}EAF$ is a trihedral angle. Since $\angle EDA = \angle FDA = 90°$, then by §168, $\angle EDF < 180°$. That is: $x < 180°$. Similarly, $y < 180°$, $z < 180°$.

(b)

4) Adding the above results: $x + y + z < 540°$.

(c)

5) In plane $N$ draw $DH \perp EF$. Draw $AH$. Then $AH \perp EF$ (§ 27).
6) $\triangle$ $AHF$ and $DHF$ are right triangles. $\angle$ $HAF$ and $HDF$ are each acute.
7) $\sin HAF = \dfrac{HF}{AF}$; $\sin HDF = \dfrac{HF}{DF}$.
8) But $DF < AF$ (Ref. 33). $\therefore\ \sin HDF > \sin HAF$. And since each of these angles is acute: $\angle HDF > \angle HAF$. Similarly: $\angle HDE > \angle HAE$.
9) $\therefore\ \angle EDF > \angle EAF$. That is, $x > \angle CAB$. In like manner we can prove that $y > \angle ABC$, $z > \angle BCA$.
10) But $\angle CAB + \angle ABC + \angle BCA = 180°$, since they are the angles of $\triangle ABC$.
11) $\therefore\ x + y + z > 180°$.

**170. Equal Polyhedral Angles.** Two polyhedral angles are said to be *equal* (or *congruent*) if one can be made to coincide with the other.

**171. Symmetric Polyhedral Angles.** If the planes of the faces of any polyhedral angle are extended through the vertex a second polyhedral angle is formed which is said to be *symmetric* with respect to the given polyhedral angle. The two polyhedral angles are also said to be symmetric with respect to each other, or simply symmetric.

For example, in Fig. 203, the planes of the faces of the trihedral angle $V\text{–}ABC$ have been produced through $V$ to form a second trihedral angle $V\text{–}A'B'C'$. By definition, these two trihedral angles, then, are symmetric. The edges $VA'$, $VB'$, $VC'$ correspond to the edges $VA$, $VB$, $VC$. It is obvious enough that the face angles of trh $\angle V\text{–}A'B'C'$ are equal respectively to the face angles of trh $\angle V\text{–}ABC$; also, the dihedral angles of the one are equal respectively to the dihedral angles of the other. All the *parts* of $V\text{–}A'B'C'$ are equal to the corresponding parts of $V\text{–}ABC$. And yet one

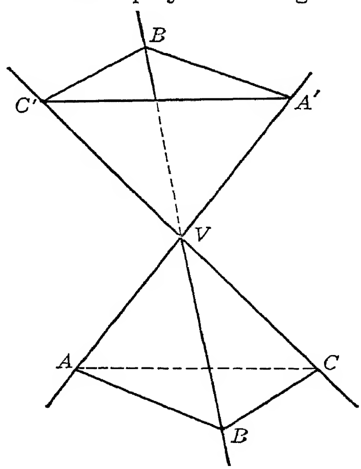



Fig. 203

trihedral angle cannot be made to coincide with the other, for the parts of one are arranged in an order which is opposite to that in which the parts of the other are arranged. In order to perceive this fact more easily, imagine that trh $\angle V\text{-}A'B'C'$ has been separated from trh $\angle V\text{-}ABC$ and placed with its vertex pointing upward. (See Fig. 204.) It is now seen that the edges $VA$, $VB$, $VC$ follow one another around in a counterclockwise direction, while the corresponding edges $VA'$, $VB'$, $VC'$ are arranged in clockwise order.

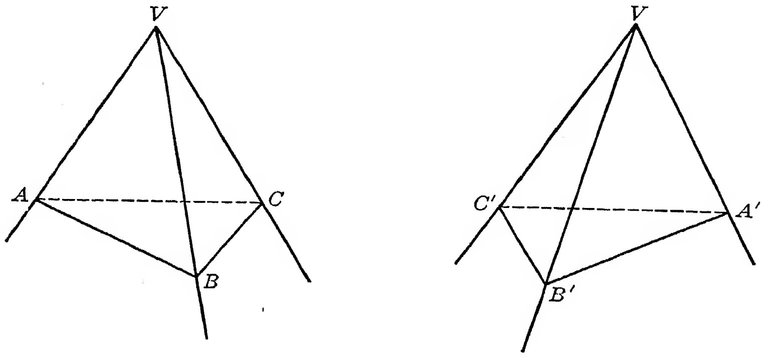



FIG. 204

Familiar instances of symmetric bodies are a pair of gloves or a pair of shoes. The two gloves (or shoes) are alike, part for part, but the corresponding parts are arranged in opposite orders. Hence the right-hand glove (or right shoe) could not possibly be substituted for the left.

**172.** If by the term "*parts of a polyhedral angle*" we imply its face angles and its dihedral angles, then from the preceding discussion we may draw the conclusions:

(a) corresponding parts of *equal* polyhedral angles are equal;
(b) corresponding parts of *symmetric* polyhedral angles are equal.

## EXERCISES

### *Group Sixteen*

**1.** Can a trihedral angle be constructed in which the face angles are respectively 60°, 40°, 100°? 160°, 130°, 170°? 120°, 120°, 120°?

**2.** Can a trihedral angle be constructed in which the dihedral angles are respectively 90°, 90°, 90°? 60°, 50°, 50°? 80°, 75°, 70°? 20°, 10°, 145°?

**3.** Two dihedral angles of a certain trihedral angle are 40° and 60°. Between what two limits (in value) does the third dihedral angle lie?

**4.** Prove that if the face angles of a trihedral angle are each 90° then each dihedral angle is also 90°.

**5.** State and prove a converse of Ex. 4.

**6.** In a trh $\angle V\text{-}ABC$, $\angle AVB = \angle CVB = 90°$. Prove that dh $\angle VA$ = dh $\angle VC = 90°$.

**7.** In a trh $\angle V\text{-}ABC$, dh $\angle VA$ = dh $\angle VC = 90°$. Prove that $\angle AVB = \angle CVB = 90°$.

**8.** Prove that two symmetric trihedral angles which have two face angles of one respectively equal to two face angles of the other can be made to coincide and are therefore equal.

**9.** Prove that two trihedral angles are equal if two face angles and the included dihedral angle of the one are respectively equal to two face angles and the included dihedral angle of the other, — corresponding parts of the two trihedral angles being arranged in like orders. (Use the method of superposition.)

**10.** Prove that two trihedral angles are equal if two dihedral angles and the included face angle of the one are respectively equal to two dihedral angles and the included face angle of the other, — corresponding parts of the two trihedral angles being arranged in like orders.

**11.** Show that in any *polyhedral* angle each dihedral angle must be less than 180°.

**173.** THEOREM 48.

Two trihedral angles are equal if the three face angles of the one are respectively equal to the three face angles of the other, — corresponding parts of the two trihedral angles being arranged in like orders.

*Given:* trh $\angle s$ $V\text{-}XYZ$, $W\text{-}KPO$
$\angle XVY = \angle KWP$, $\angle YVZ$ $\angle PWO$, $\angle ZVX = \angle OWK$.

*Prove:* trh $\angle V$ = trh $\angle W$.

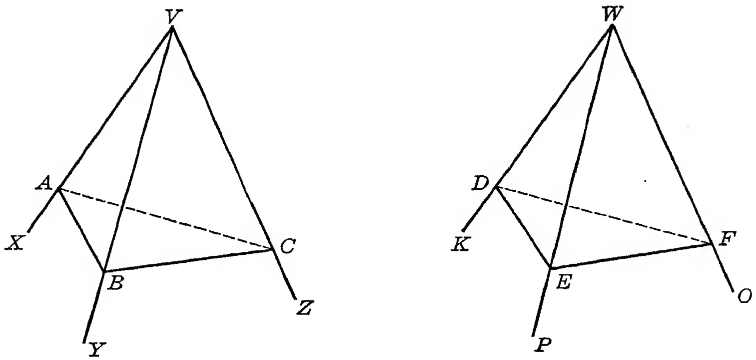



FIG. 205

If we can prove dh $\angle VX$ = dh $\angle WK$, we can prove $V = W$ by superposition. (Cf. Ex. 9, Group 16.)

1) On $VX$ and $WK$ respectively take $VA = WD$. Through $A$ and $D$ respectively draw planes $S$ and $T$ perpendicular to $VX$ and $WK$.

2) $\therefore \angle BAC$ is the plane angle of dh $\angle VX$, and $\angle EDF$ is the plane angle of dh $\angle WK$.
3) rt $\triangle VAB \cong$ rt $\triangle WDE$; rt $\triangle VAC \cong$ rt $\triangle WDF$ (Ref. 2).
4) $\therefore AB = DE$ and $AC = DF$. Also $VB = WE$ and $VC = WF$.
5) Show that $\triangle BVC \cong \triangle EWF$.
6) $\therefore BC = EF$.
7) Now show $\triangle ABC \cong \triangle DEF$ (SSS).
8) $\therefore \angle BAC = \angle EDF$.
9) $\therefore$ dh $\angle VX =$ dh $\angle WK$.
10) Use the method of Ex. 9, Group 16 (superposition) to show that trh $\angle V$ = trh $\angle W$.

(In case the corresponding parts of the two trihedral angles are arranged in opposite orders, of course the two trihedral angles are then symmetric instead of equal.)

**174. Polyhedron.** A polyhedron is a solid bounded by portions of planes. (Prisms and pyramids already studied are types of polyhedrons.)

The *edges* of a polyhedron are the lines of intersection of the faces.

The *vertices* are the vertices of the polygons which form the faces.

The *area* of a polyhedron is the sum of the areas of its faces.

At each vertex, obviously, there is a polyhedral angle of some sort.

A polyhedron is *convex* or *concave* according as any plane section of it is a convex or a concave polygon. Only the convex type is considered here.

A polyhedron is *regular* if all its polyhedral angles are equal, and if all its faces are congruent regular polygons.

**175.** THEOREM 49.

There cannot be more than *five* different types of regular polyhedrons.

The face angles of each polyhedral angle of the polyhedron are the vertex angles of the regular polygons which form the faces of the polyhedron. The number of different types of regular polyhedrons possible, therefore, depends upon (a) the *number* of regular polygons which can be grouped about a common vertex to form each polyhedral angle, and (b) the *types* of polygons used for this purpose.

By § 168, the sum of the face angles of each polyhedral angle must be less than 360°. Therefore, we may draw the following conclusions:

1) If *equilateral triangles* are used as faces, a polyhedral angle can be formed by using *three* or *four* or *five* of these triangles but no more, since the sum of the face angles must be kept less than 360°. Hence there are no more than *three* types of regular polyhedrons having equilateral triangles as faces.
2) If *squares* are used as faces, a polyhedral angle can be formed by using *three* squares and no more than three. Therefore, there can be no more than *one* type of regular polyhedron having squares as faces.
3) If *regular pentagons* are used as faces, a polyhedral angle can be formed by

using *three* of these and no more. Hence there can be no more than *one* type of regular polyhedron having regular pentagons as faces.

4) Regular hexagons or regular polygons of any greater number of sides cannot be used, since the grouping of three or more about a common vertex would cause the sum of the face angles of each polyhedral angle either to equal or to exceed 360°.

5) Therefore, there are no more than *five* different types of regular polyhedrons.

Note carefully that we have not attempted to prove that there *are* precisely five types of regular polyhedrons. It is true that there are exactly five types; but we have shown merely that there cannot be *more* than five types. The proof of the existence of the five types is beyond the scope of this book.

| Types of Faces | Number of Polygons Used to Form Each Polyhedral Angle | Sum of Face Angles of Each Polyhedral Angle | Total Number of Faces | Name of Polyhedron |
|---|---|---|---|---|
| Equilat. △ | 3 | 180° | 4 | Tetrahedron |
| Equilat. △ | 4 | 240° | 8 | Octahedron |
| Equilat. △ | 5 | 300° | 20 | Icosahedron |
| Square | 3 | 270° | 6 | Hexahedron |
| Reg. Pentagon | 3 | 324° | 12 | Dodecahedron |

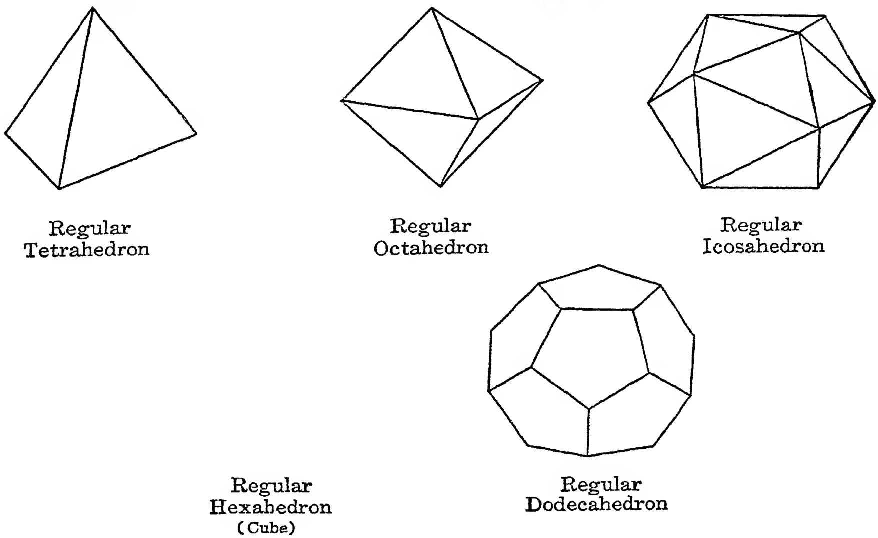



Fig. 206

In order to study the five different types of regular polyhedrons more easily it may be helpful to construct models of them from heavy paper or card board. The diagrams which follow are patterns for these solids. Fold along dotted lines.

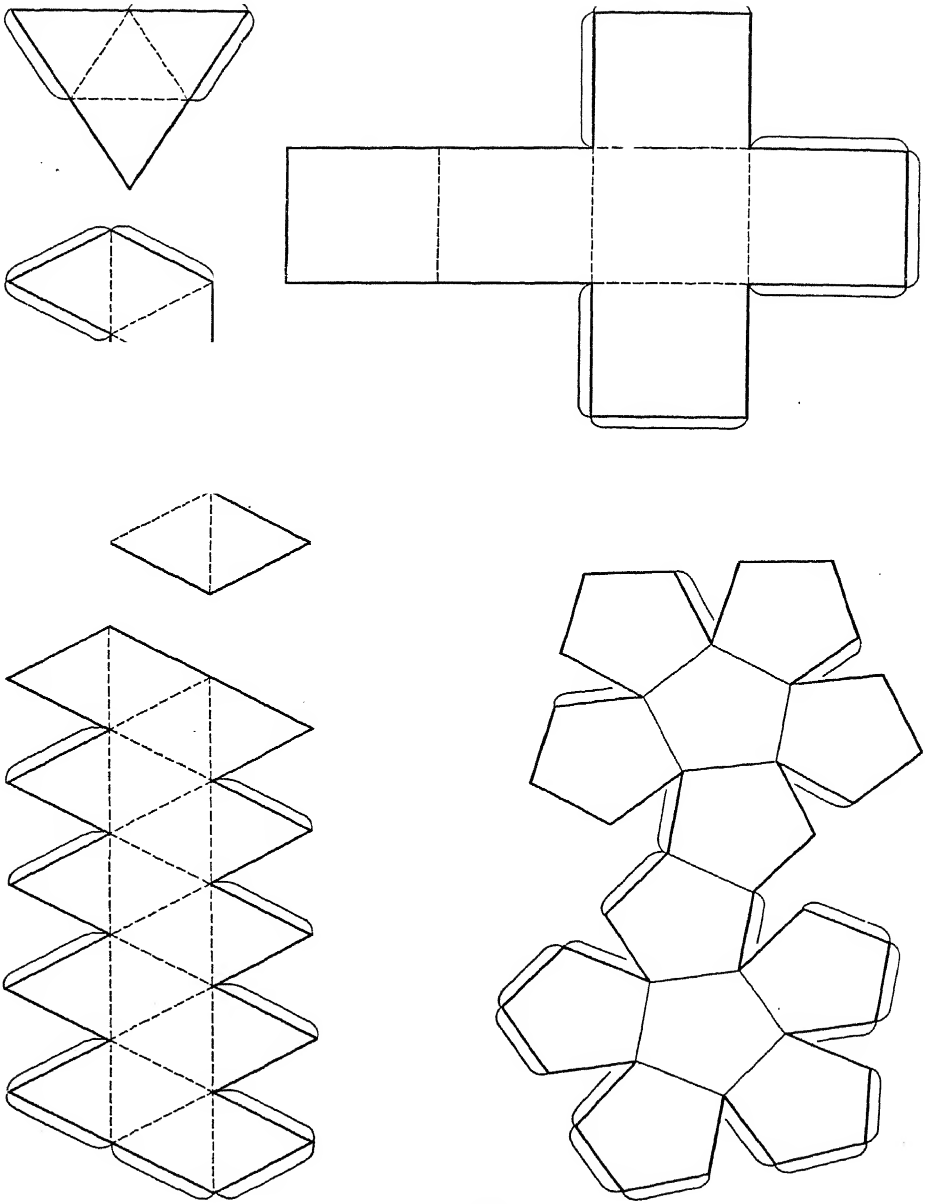


FIG. 207

**176. Similar Polyhedrons.** Similar polyhedrons are those whose corresponding polyhedral angles are equal and whose corresponding faces are similar polygons which are similarly placed.

**177. Exercise.** Prove that if two tetrahedrons have a trihedral angle in common their volumes are to each other as the products of the edges including the common trihedral angle.

Let the two tetrahedrons be $T\text{–}ABC$ and $T\text{–}DEF$, having trh $\angle T$ in common.
Let $TA = b$, $TB = a$, $TC = x$, $TD = e$, $TE = d$, $TF = y$.
Let $\angle BTA = \theta$.

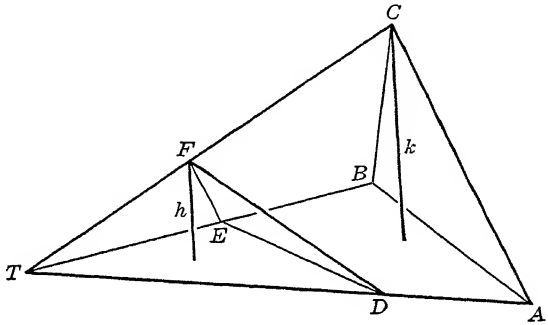



FIG. 208

*Prove:* $\dfrac{v_1}{v_2} = \dfrac{bax}{edy}$

From $F$ and $C$, respectively, draw $h$ and $k$ perpendicular to the plane $BTA$.

Show that $\dfrac{k}{h} = \dfrac{x}{y}$.

Area $\triangle BTA = \frac{1}{2}ba \sin \theta$; area $\triangle ETD = \frac{1}{2}ed \sin \theta$.

Using $h$ and $k$ as altitudes express the volumes of the two tetrahedrons.

Divide one result by the other.

**178. Exercise.** Prove that the volumes of two similar tetrahedrons are to each other as the cubes of any two corresponding edges.

**179.** THEOREM 50.

> The volumes of two similar polyhedrons are to each other as the cubes of any two corresponding edges or as the cubes of any two corresponding lines.

The following is not designed to be a complete and rigorous proof of the theorem. The intention is to make the truth of the theorem acceptable by indicating certain ideas upon which a proof could be based if the necessary logical background were at hand.

Let the similar polyhedrons be $D\text{–}ABPC\text{–}E$ and $W\text{–}XYQZ\text{–}T$. Assume that the solids are similarly placed.

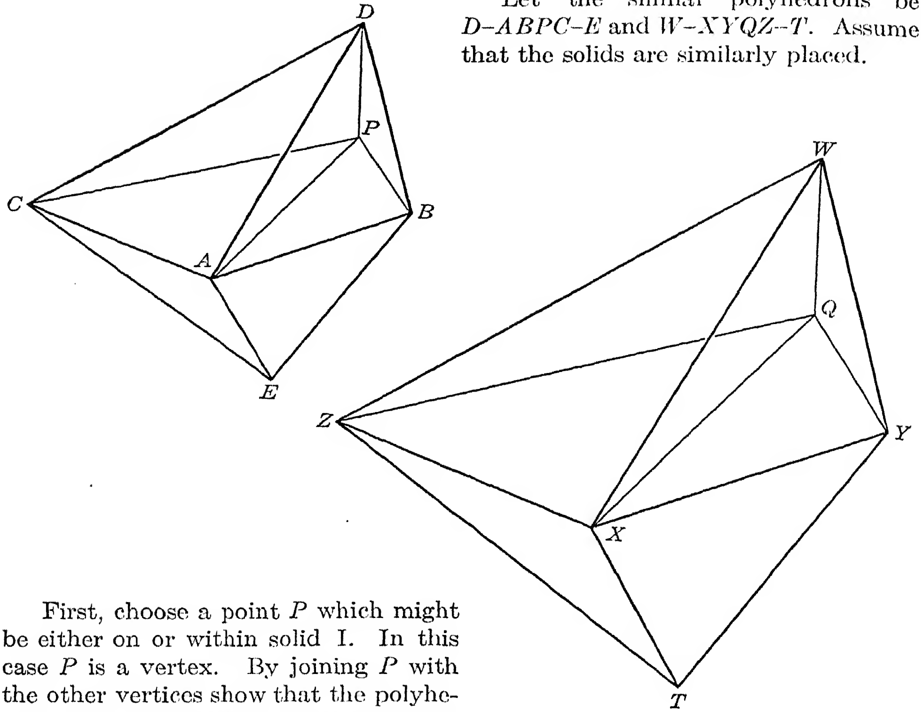



FIG. 209

First, choose a point $P$ which might be either on or within solid I. In this case $P$ is a vertex. By joining $P$ with the other vertices show that the polyhedron can be resolved into $n$ tetrahedrons having a common vertex $P$. Secondly, assume that in solid II a point $Q$ corresponding to $P$ can be found. Here $Q$ is a vertex corresponding to $P$. Assume that solid II can be resolved into $n$ tetrahedrons having the common vertex $Q$ and similar to the tetrahedrons of solid I.

Consider a pair of these similar tetrahedrons: $P\text{–}ADC$ and $Q\text{–}XWZ$.

From § 178, $\dfrac{P\text{–}ADC}{Q\text{–}XWZ} = \left(\dfrac{AC}{XZ}\right)^3 = \left(\dfrac{AD}{XW}\right)^3$

Similarly, $\dfrac{P\text{–}ABD}{Q\text{–}XYW} = \left(\dfrac{AD}{XW}\right)^3 = \left(\dfrac{AB}{XY}\right)^3$

$$\frac{P\text{–}ADC}{Q\text{–}XWZ} = \frac{P\text{–}ABD}{Q\text{–}XYW} = \left(\frac{AB}{XY}\right)^3$$

Show that $\dfrac{P\text{–}ADC + P\text{–}ABD}{Q\text{–}XWZ + Q\text{–}XYW} = \left(\dfrac{AB}{XY}\right)^3$

Continue this process until you have:

$$\frac{\text{Sum of tetrahedrons of I}}{\text{Sum of tetrahedrons of II}} = \left(\frac{AB}{XY}\right)^3 \quad \text{or} \quad \frac{\text{I}}{\text{II}} = \left(\frac{AB}{XY}\right)^3$$

**180.** Corollary A (Th. 50).

| |
|---|
| The areas of two similar polyhedrons are to each other as the squares of any two corresponding edges or as the squares of any two corresponding lines. |

# *Chapter Twelve*

## SPHERES. (GENERAL PROPERTIES)

**181. Sphere.** A *sphere* is a solid all points of whose surface are at a constant distance from a fixed point within the solid.

Thus, a *spherical surface* is the locus of points which are at a constant distance from a fixed point.

The *center* of a sphere is the fixed point mentioned above.

The *radius* is the constant distance.

A *diameter* is a line-segment passing through the center of the sphere and terminated by the surface. A diameter is equivalent to the sum of two radii.

The *area* of a sphere is the area of the spherical surface.

A point is said to be within, on, or outside a sphere according as its distance from the center is less than, equal to, or greater than the length of a radius.

The *distance* from a given point to the surface of a given sphere is always measured along the straight line connecting the given point with the center of the sphere.

A sphere may be generated by revolving a semicircle through 360° about its diameter, or by revolving a complete circle through 180° about any diameter.

As in the case of circles, when there is no ambiguity a sphere may be designated by naming its center.

**182.** THEOREM 51.

| Any plane section of a sphere is a circle. |
|---|

*Given:* Plane $M$ cutting sphere $O$. $M$ cuts the spherical surface in some curve $c$.

*Prove:* $c$ is a circle.

1) Draw $OP \perp M$, cutting $M$ at $P$.
2) Choose any number of points $A, B, C, \ldots$ on $c$. Draw $AP, BP, \ldots, AO, BO, \ldots$
3) Show that $AP = BP = \cdots$ by congruent triangles.
4) Show that $c$ satisfies the definition of circle.
5) If $M$ passes through $O$, why is $c$ obviously a circle? What is its radius?

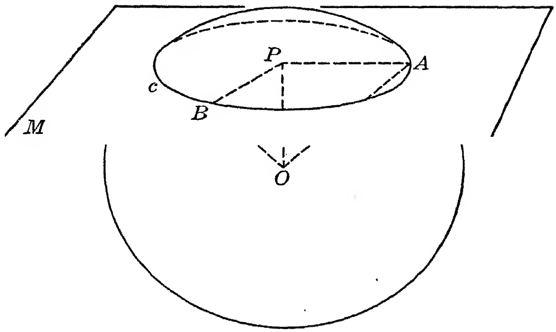



FIG. 210

**183. Circles of a Sphere.** A circle formed by the intersection of a plane with a sphere is called a *circle of the sphere*. If the intersecting plane passes through the center of the sphere the circle thus formed has its center at the center of the sphere and has the radius of the sphere as its own radius. Such a circle is called a *great circle*. All circles of a sphere other than great circles are often called *small circles*.

The following facts relating to circles of a sphere are easily deduced and may be done as exercises.

A. Three points on the surface of a sphere determine one and only one circle of the sphere.

B. Two points (not the extremities of a diameter) on the surface of a sphere determine one and only one great circle.

C. A diameter of a sphere which is perpendicular to the plane of any circle of the sphere passes through the center of that circle.

D. A diameter of a sphere which passes through the center of a small circle is perpendicular to the plane of that circle.

E. If two small circles of a given sphere are equal they are equidistant from the center of the sphere, and conversely.

F. If two small circles of a given sphere are unequal the larger circle is nearer to the center of the sphere than is the smaller, and conversely.

G. A great circle of a given sphere is the largest circle of that sphere.

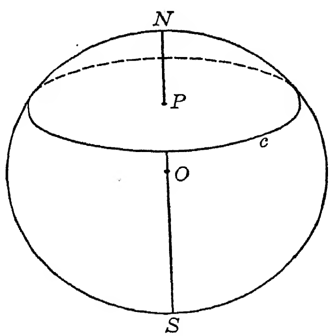



FIG. 211

**184. Axis and Pole.** The *axis* of a circle of a sphere is that diameter of the sphere which passes through the center of that circle. By § 183, the axis must be perpendicular to the plane of the circle. In Fig. 211, $NS$ is the axis of the circle $c$.

The *poles* of a circle of a sphere are the ex-

tremities of its axis. In Fig. 211, $N$ and $S$ are the poles of circle $c$; $N$ is the *nearer pole*, $S$ the *farther pole*.

**185. Spherical Distance.** If $A$ and $B$ are any two points on the surface of a given sphere the distance between $A$ and $B$ measured along the spherical surface is an arc of the great circle determined by $A$ and $B$; and the arc chosen is one which is never greater than half a great circle. This distance is called the *spherical distance* between $A$ and $B$.

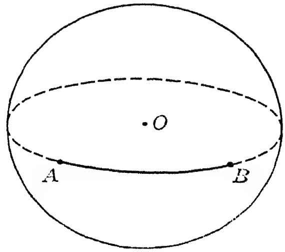


FIG. 212

**186. Quadrant.** A quadrant is one-fourth the circumference of a great circle.

**187.** THEOREM 52.

The spherical distance from any point on a given circle of a given sphere to a specified pole of that circle is constant.

*Given:* $c$, a circle of sphere $O$.
$A$, any point on $c$.
$N$ and $S$, the poles of $c$.

*Prove:* Spherical distance $\overset{\frown}{AN}$ is constant.

1) Choose a second point $B$ on $c$. Draw the two great circles determined by $A$ and $N$, and by $B$ and $N$, respectively. Draw $OA$, $AP$, $OB$, $BP$.
2) $\triangle OAP \cong \triangle OBP$. $\therefore$ $\angle AOP = \angle BOP$.
3) $\therefore$ $\overset{\frown}{AN} = \overset{\frown}{BN}$.
4) Similarly, choose other points $C$, $D$, etc., on $c$ and show that $\overset{\frown}{AN} = \overset{\frown}{BN} = \overset{\frown}{CN} = \overset{\frown}{DN} = \cdots$
5) In like manner we can show that spherical distance $\overset{\frown}{AS}$ is constant.

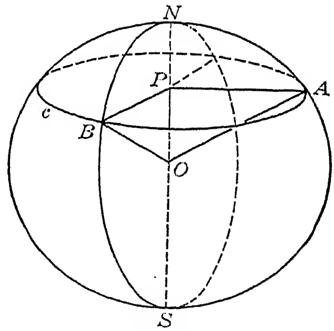


FIG. 213

**188. Polar Distance.** The spherical distance from any point on a given circle of a sphere to the *nearer* pole is called the *polar distance* of that circle.

**189.** THEOREM 53.

If a point $P$ on a given sphere is at a quadrant's distance from each of two other points $A$ and $B$ (not the extremities of a diameter) also on the sphere, then $P$ is a pole of the great circle determined by $A$ and $B$.

*Given:* $\overset{\frown}{PA} = \overset{\frown}{PB} = 90°$.
$A$ and $B$ are not the ends of a diameter.
$c$ is the great circle determined by $A$ and $B$.

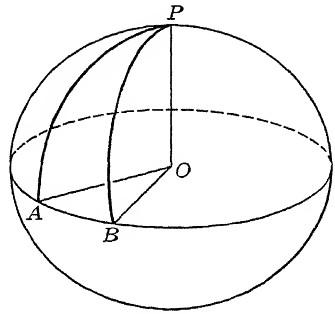



FIG. 214

*Prove:* $P$ is a pole of circle $c$.

1) Draw $PO$, $OA$, $OB$.
2) Show that $\angle POA = \angle POB = 90°$ (Ref. 73).
3) Show that $PO$ is the axis of $c$, and hence that $P$ is a pole of $c$.

## EXERCISES

*Group Seventeen*

**1.** Why do § 183–$B$ and § 189 break down in case the "two points" mentioned *are* the extremities of a diameter of the sphere?

**2.** What is the polar distance of any great circle?

**3.** On a given sphere equal circles have equal polar distances, and conversely. Prove.

**4.** If the planes of two circles of a given sphere are parallel, these circles have the same axis and the same poles. Prove.

**5.** How many points are necessary to determine a sphere? Describe possible arrangements of these points.

**6.** $A$ and $B$ are the extremities of a diameter of a sphere, and $P$ is any third point on the surface. Show that $\angle APB$ is a right angle.

**7.** $A$ and $B$ are any two points on a sphere. Draw the chord $AB$ and the plane $S$ which bisects $AB$ perpendicularly. Prove that $S$ must pass through the center of the sphere.

**8.** If the axis of one great circle is taken as the diameter of a second great circle on the same sphere, the planes of the two great circles are perpendicular to each other. Prove.

**9.** If the planes of two great circles of a sphere are perpendicular to each other, either circle must contain the poles of the other. Prove.

**10.** The diameter of a sphere is 20″. The radius of a small circle is 8″. How far from the center of the sphere is the plane of this circle?

**11.** The plane of a small circle of a sphere is 15″ from the center. If the radius of the sphere is 17″, what is the radius of the given circle?

**12.** The diameter of a sphere is 30 cm. A plane bisects a radius perpendicularly. What is the area of the section determined by this plane?

**13.** The radius of a sphere is 24 cm. How far from the center of the sphere must a plane be drawn in order to determine a small circle the area of which will be one-fourth the area of a great circle? One-half the area of a great circle?

**14.** On a sphere of radius 12″ the polar distance of a small circle is an arc of 60°. Find the polar distance in inches.

**15.** In Ex. 14 find the radius of the small circle.

**16.** What is the locus of points which are at the same time $d$ inches from a fixed point $A$ and equidistant from two other points $B$ and $C$? Is the locus always possible?

**17.** What is the locus of points which are 2″ from the surface of a sphere the radius of which is 8″?

**18.** The radius of a sphere is 6″. Two points $A$ and $B$ are each 11″ from the center of the sphere. What is the locus of points which are at the same time 2″ from the surface of the sphere and equidistant from $A$ and $B$?

**19.** $A$ and $B$ are two points equidistant from the surface of a given sphere, and the two points lie outside the sphere. Prove that the plane which bisects perpendicularly the line-segment $AB$ passes through the center of the sphere. Is there any material difference resulting in case $A$ and $B$ both lie within the sphere?

**20.** The radius of a sphere is 4″. $O$ is the center. Point $A$ is 12″ from $O$. What is the locus of points which are at the same time 6″ from the spherical surface and equidistant from $O$ and $A$?

**190.** THEOREM 54.

If two spheres intersect each other the intersection of their surfaces is a circle. (The plane of this circle is perpendicular to the line of centers, and the center of the circle lies on the line of centers.)

*Given:* Spheres $A$ and $B$ intersecting each other.
$a$ and $b$ are the respective radii.

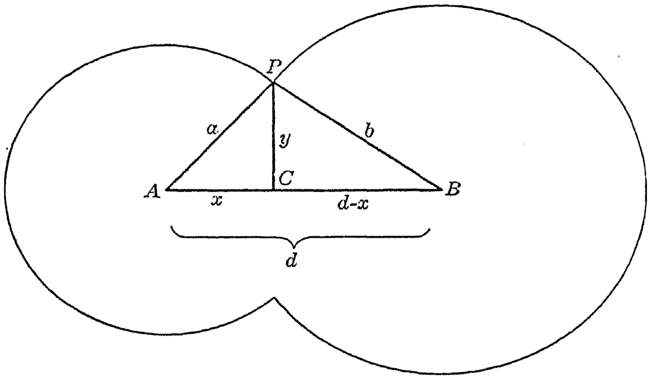



FIG. 215

*Prove:* Intersection of the surfaces of $A$ and $B$ is a circle.

(a) Let $P$ be any point on the intersection of the surfaces.

1) Draw $PA$, $PB$, $AB$; draw $PC \perp AB$. Then $PA = a$, $PB = b$. Let $AB = d$, $PC = y$, $AC = x$; then $CB = d - x$.

2) $y^2 = a^2 - x^2$; also, $y^2 = b^2 - (d - x)^2 = b^2 - d^2 + 2dx - x^2$ (Ref. 13).

3) Hence, $a^2 - x^2 = b^2 - d^2 + 2dx - x^2$ (Ax. 1).

4) $\therefore \quad x = \dfrac{a^2 - b^2 + d^2}{2d}$ and $y = \sqrt{a^2 - \left(\dfrac{a^2 - b^2 + d^2}{2d}\right)^2}$

5) In step (4) the right members are each constant since $a$, $b$, $d$ are each constant. $\therefore$ $x$ and $y$ must each be constant, regardless of the selection of $P$ (on the intersection) in the beginning.

6) $\therefore$ $P$ must lie on a circle $t$ which has a constant radius $PC$ and a fixed center $C$ on $AB$.

(b) Let $Q$ be any point on the circle $t$ found in part (a).

7) Draw $QA$ and $QB$. Show that $QA = a$ and that $QB = b$, and hence that $Q$ lies both on sphere $A$ and sphere $B$. $\therefore$ $Q$ must lie on the intersection of the surfaces of $A$ and $B$.

8) From (a): any point common to the surfaces of $A$ and $B$ lies on the circle $t$. From (b): any point on $t$ is common to $A$ and $B$. $\therefore$ $t$ is the *locus* of points which are *common* to the two spherical surfaces.

9) $\therefore$ the circle with radius $PC$ and center $C$ must be the intersection of the surfaces of the two spheres $A$ and $B$.

**191.** THEOREM 55.

The spherical distance between two given points on a sphere is the shortest distance between the two points, — distances being measured along the spherical surface.

*Given:* $A$ and $B$, any two points on sphere $O$. $\widehat{AB}$ is the spherical distance.

*Prove:* $\widehat{AB}$ is the shortest path from $A$ to $B$ along the spherical surface.

1) Let $P$ be any point on $\widehat{AB}$. Let curve $x$ be some path from $A$ to $B$ other than $\widehat{AB}$ along the surface (Fig. 216–$A$).

2) $x$ cannot be the shortest path from $A$ to $B$, for we can find a shorter path as follows (Figs. 216–$B$, 216–$C$):
   (a) With $A$ and $B$ as poles and $\widehat{AP}$ and $\widehat{BP}$ as polar distances draw two small circles cutting $x$ at $C$ and $D$. Let $\widehat{AC} = y$, $\widehat{DB} = z$, $\widehat{CD} = w$.
   (b) Now rotate circles $A$ and $B$ about their centers so that points $C$ and $D$ come together at $P$. Curves $y$ and $z$ will have changed their positions but not their lengths. Simultaneously, the piece $w$ will have vanished; hence the original path $ACDB$ is now lessened by the amount $w$.
   (c) $\therefore$ path $(y + z)$ is shorter than path $(y + w + z)$; hence $x$ is not the shortest path from $A$ to $B$.

3) We now have the conclusion: Any path which does *not* go through $P$ cannot be the shortest path. We are assuming that there *is* a shortest path. Therefore, the shortest path from $A$ to $B$, — whatever it is — *does* go through $P$.

4) But $P$ is *any* point on $\widehat{AB}$. If any other points $Q$, $R$, $S$, $T$, etc., are chosen and treated in the same way, we find that the shortest path must pass

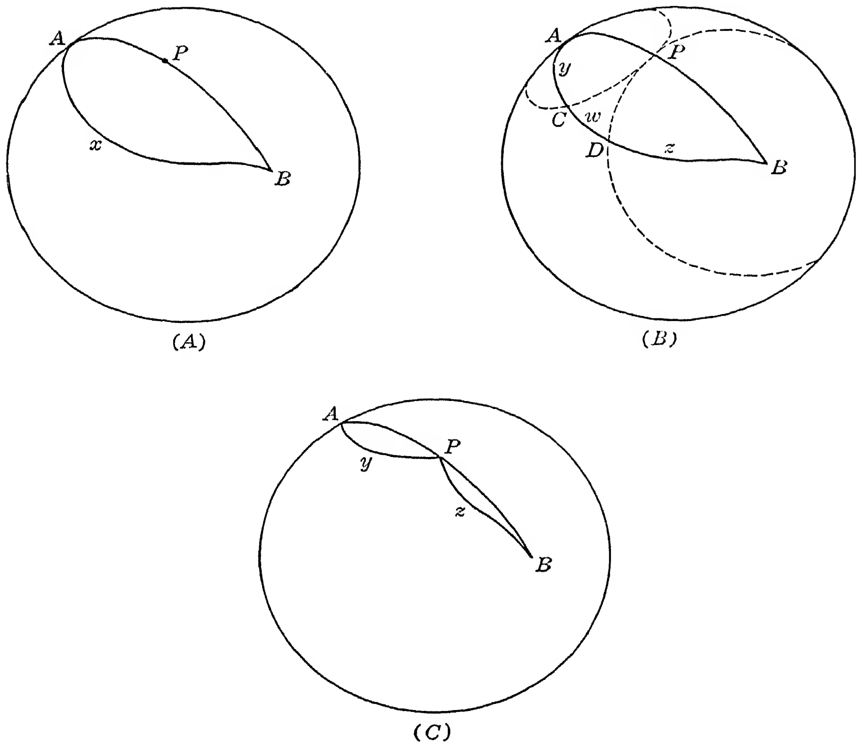



Fig. 216

through $P$, $Q$, $R$, $S$, $T$, . . . Hence by continuing this process indefinitely we find that the shortest path from $A$ to $B$ must pass through *all* the points of $\widehat{AB}$.

5) But *all* the points of $\widehat{AB}$ constitute the arc $AB$ itself.

6) $\therefore$ the shortest path from $A$ to $B$ along the surface of the sphere is the spherical distance $\widehat{AB}$.

**192. Tangents to a Sphere.** A *plane* and a *sphere* are *tangent* to each other if the two have one and only one point in common. A *line* and a *sphere* are *tangent* to each other if the two have one and only one point in common.

**193. Tangent Spheres.** Two *spheres* are *tangent* to each other if the two are tangent to the same plane at the same point. The terms "tangent internally" and "tangent externally" are used as in the cases of tangent circles in Plane Geometry. It can be deduced that the surfaces of two tangent spheres have one and only one point in common.

**194.** THEOREM 56.

If a plane is perpendicular to a radius of a sphere at the outer end of that radius, the plane is tangent to the sphere. Conversely, if a plane is tangent to a sphere, it is perpendicular to the radius which is drawn to the point of contact.

PART 1

1) Let $O$ be the center of the sphere, $OT$ the given radius, plane $M \perp OT$ at $T$ (Fig. 217).
2) In $M$ choose any point $A$ (other than $T$), and draw $OA$.
3) Show that $OA > OT$ and hence that point $A$ must lie outside the sphere.
4) Hence, show that $T$ is the only point which $M$ and the sphere have in common.

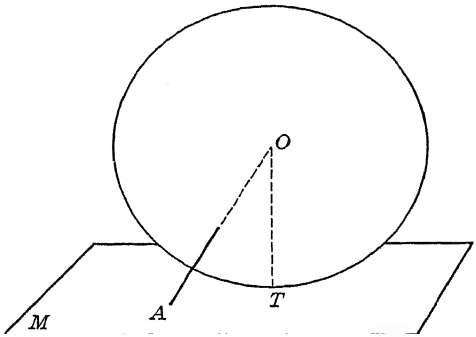


FIG. 217

PART 2

(Left as an exercise)

**195. Inscribed and Circumscribed Polyhedrons.** A polyhedron is *inscribed* in a sphere (or the sphere is circumscribed about the polyhedron) if the vertices of the polyhedron lie on the surface of the sphere.

A polyhedron is *circumscribed* about a sphere (or the sphere is inscribed in the polyhedron) if the faces of the polyhedron are each tangent to the sphere.

**196.** THEOREM 57.

A sphere can be inscribed in or circumscribed about
(a) any tetrahedron;
(b) any regular polyhedron.

For a proof of (a) see Exs. 24, 25 of Group Twelve; §§ 134, 135. Assume (b) without proof.

**EXERCISES**

*Group Eighteen*

**1.** If a plane is tangent to a sphere, a line perpendicular to the plane at the point of contact must pass through the center of the sphere. Prove.

**2.** If a plane is tangent to a sphere, a line from the center of the sphere perpendicular to this plane must pass through the point of contact. Prove.

3. At a given point on a given sphere there is one and only one plane which is tangent to the sphere. Prove.

4. A plane $M$ is tangent to a sphere $O$ at a point $A$. Prove that any line in $M$ containing $A$ is tangent to the sphere.

5. Two lines $x$ and $y$ are tangent to a sphere $O$ at point $A$. Prove that the plane of $x$ and $y$ is tangent to the sphere.

6. Two planes $M$ and $N$ intersecting each other in a line $x$ are tangent to a sphere $O$ at points $A$ and $B$, respectively. Prove that the plane of $A$, $O$, $B$ is perpendicular to $x$.

7. The radii of two spheres are respectively $5''$ and $\sqrt{34}''$, and the centers are $9''$ apart. Find the circumference of the circle of intersection.

8. A sphere $O$ is tangent to two parallel planes $M$ and $N$ at points $A$ and $B$, respectively. Prove that $A$, $O$, $B$ are collinear.

9. Points $A$ and $B$ are $9''$ apart. What is the locus of points which are at the same time $5''$ from $A$ and $2\sqrt{13}''$ from $B$?

10. The radii of two concentric spheres are respectively $10''$ and $13''$. A plane $S$ bisects perpendicularly a radius of the smaller sphere. Find the area of that portion of $S$ which is within the larger sphere but outside the smaller sphere.

11. The radius of a right circular cone is $4''$ and the altitude is $4\sqrt{3}''$. A sphere is inscribed in the cone. (A sphere is inscribed in a cone if it is tangent to the elements and to the plane of the base.) Find the radius of the sphere.

12. In Ex. 11 what is the locus of points which are common to the spherical surface and the lateral surface of the cone? Find the actual length of this locus.

13. Find the radii of the spheres which are respectively inscribed in and circumscribed about a regular tetrahedron one edge of which is $6''$.

14. The radius of a sphere is $12''$. Find the volume of the regular tetrahedron which can be inscribed in the sphere.

15. Three equal spheres of radius 10 cm. are tangent to one another; and a fourth sphere equal to each of the first three is tangent to them. Find the altitude of the regular tetrahedron which has the centers of these spheres as its vertices.

16. In Ex. 15 find the radius of the sphere which can be inscribed in the tetrahedron.

17. Point $A$ is outside a sphere $O$. Prove that all the lines which can be drawn from $A$ tangent to the sphere are equal. What is the locus of the points of tangency?

18. What is the locus of the centers of all the spheres which are tangent to a plane $M$ at a common point $A$? Prove it.

19. $c$ is a circle. Any number of spheres is drawn so that their surfaces all contain circle $c$. What is the locus of the centers of these spheres?

20. If two spheres are tangent to each other their line of centers passes through their point of contact. Prove.

21. Prove that a sphere may be inscribed in or circumscribed about any given cube.

**22.** Find the radii of the spheres which are respectively inscribed in and circumscribed about a cube the edge of which is $e$.

**23.** The radius of a sphere is $r$. Find the edge of the inscribed cube; the edge of the circumscribed cube.

**24.** The radius of a sphere is 10′. A right circular cone has its vertex on the surface of the sphere and its base is tangent to the sphere. The radius of the cone is 10′. Show that the intersection of the spherical surface with the lateral surface of the cone is a circle. Find the radius of this circle.

**25.** A regular square pyramid whose altitude is 4″ is inscribed in a sphere of radius 3″. Find the basal area of the pyramid.

**26.** A sphere is circumscribed about an equilateral cone. If the radius of the cone is 12″, what is the radius of the sphere? (The angle of an equilateral cone is 60°.)

**27.** Do Ex. 26 assuming that the cone is circumscribed about the sphere.

**28.** Find the radius of the sphere inscribed in a regular tetrahedron whose edge is 24′.

**29.** $\triangle ABC$ is right-angled at $C$, and $AB$ is fixed in length and position. What is the locus of point $C$?

**30.** Two planes $M$ and $N$ meet in a line $x$. What is the locus of the centers of the spheres which are tangent to $M$ and $N$?

**197. Points on the Surface of the Earth.** In Plane Analytical Geometry we customarily locate a point in a plane by giving its coördinates, that is, its distances from two fixed perpendicular lines called coördinate axes. Any point on the surface of the Earth, — which for practical purposes is assumed to be a perfect sphere of radius approximately 4000 miles — is located by referring it to two fixed great circles.

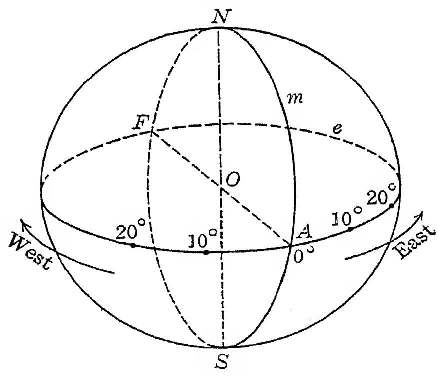


FIG. 218

One of these great circles is the equator ($e$ in Fig. 218). On $e$ let $A$ be some fixed point. Let $N$ and $S$ be the poles of $e$. Draw the great circle $m$ determined by the points $N$, $A$, $S$. Any great circle through $N$ and $S$ is called a *meridian circle;* and the half of one of these circles included between $N$ and $S$ is usually called a *meridian.* On the Earth circle $m$ is chosen so that it passes through Greenwich; and the meridian containing Greenwich is the *Meridian of Greenwich* or the *prime meridian.* The equator and the Meridian of Greenwich are the two lines to which any point on the Earth's surface is referred for location. $NS$ is known as the *axis* of the Earth; $N$ and $S$ are respectively the *north pole* and *south pole.* As you view the diagram (Fig. 218) a direction to your right from $A$ along $e$ is *east;* the opposite direction along $e$ is *west.*

*Longitude.* Commencing at $A$ and in each direction along $e$, divide the semicircles $AF$ into one-degree arcs. Through the points of division draw meridian circles (Fig. 219).

If $\widehat{AB} = 20°$, any point $P$ on $\widehat{NBS}$ is said to have a *longitude* of 20° east. Similarly, if $\widehat{AC} = 42°$ any point $Q$ on $\widehat{NCS}$ has a *longitude* of 42° west. *Longitude* is always given with reference to the arc $NAS$ which passes through Greenwich.

What is the greatest number of degrees of longitude which a point can have?

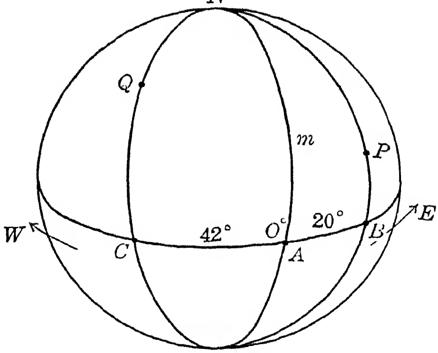



FIG. 219

*Latitude.* Select any meridian circle, one-half of which intersects the equator, say, at a point $D$ (Fig. 220). Divide the two quadrants $DN$ and $DS$ into one-degree arcs commencing at $D$. Through these points of division draw circles of the sphere whose planes are parallel to the plane of the equator. Each of these circles is a *circle of latitude* or a *parallel of latitude.*

If $\widehat{DE} = 15°$, any point on the small circle $u$ through $E$ is said to have a *latitude* of 15° north. If $\widehat{DH} = 76° \, 8'$, any point on the circle $v$ through $H$ has a *latitude* of 76° 8′ south. *Latitude* is always given with reference to the equator.

What is the greatest number of degrees of latitude which a point on the Earth's surface can have?

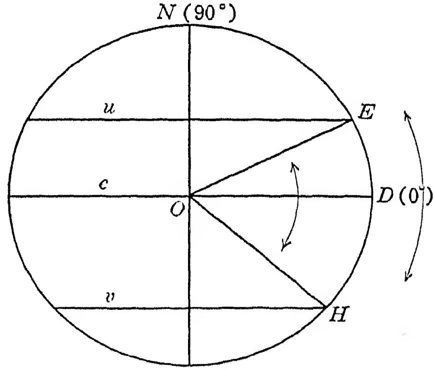



FIG. 220

Obviously, if both the latitude and longitude of a point are known, the position of the point on the Earth's surface is definitely determined.

*Nautical Mile and Knot.* A *nautical mile* is defined as the length of one minute of arc along the equator, that is, the length of one minute of arc of a great circle of the Earth. A nautical mile is approximately 6080 feet or 1.15 ordinary land miles.

A *knot* is a unit indicating rate of speed and means *one nautical mile per hour.* Thus if a ship sails at the rate of 15 knots, it is moving at the rate of 15 nautical miles per hour.

## EXERCISES

*Group Nineteen*

1. Draw a diagram in which the equator and the Meridian of Greenwich are indicated. On your diagram indicate points having the following positions: lat. 30° S, long. 62° E; lat. 90° N, long. 20° W; lat. 45° S, long. 0°; lat. 76° S, long. 180°; lat. 0°, long. 30° W; lat. 43° N, long. 70° W.

2. Find the number of nautical miles between the north pole and any point on the equator.

3. Find the number of nautical miles between the north pole and a point $P$ (lat. 19° 15′ S, long. 84° W.)

4. Given: $A$ (lat. 47° 10′ N, long. 72° W), $B$ (lat. 20° 30′ N, long. 72° W). Find the spherical distance $AB$ in nautical miles.

5. Given: $A$ (lat. 0°, long. 8° E), $B$ (lat. 0°, long. 46° 32′ W). Find the spherical distance $AB$ in nautical miles.

6. A ship steaming due north at 20 knots is at a position (lat. 19° 12′ N, long. 50° W) at 6:00 A.M. on June 20. Find the ship's position at 12:00 noon on June 22.

7. Two ships A and B are at points $P$ (lat. 48° 10′ N, long. 32° W) and $Q$ (lat. 22° S, long. 32° W), respectively. A can average 20 knots and B can average 28 knots. If A and B leave their given positions at 8:00 A.M. on September 10 and travel toward each other, when and where will they meet? Answer to the nearest minute.

# *Chapter Thirteen*

## SPHERICAL ANGLES. SPHERICAL TRIANGLES

**198. Spherical Angle.** A spherical angle is a figure on the surface of a sphere composed of two arcs of great circles emanating from the same point. The terms "vertex" and "sides" are used as in the case of ordinary plane angles.

**199. Measure of a Spherical Angle.** Let $ABC$ be any spherical angle (Fig. 221). At $B$ draw $BD$ and $BE$ tangent to $\overset{\frown}{BA}$ and $\overset{\frown}{BC}$, respectively. The *measure* of sph $\angle ABC$ is defined as the measure of $\angle DBE$, — the angle between the two tangents.

Thus, if $\angle DBE = 35°$, then sph $\angle ABC = 35°$.

A *right spherical angle* is a spherical angle of 90°.

An *acute* spherical angle is less than 90°; an *obtuse* spherical angle is greater than 90° and less than 180°.

Two great circle arcs are *perpendicular* if they meet to form a right spherical angle.

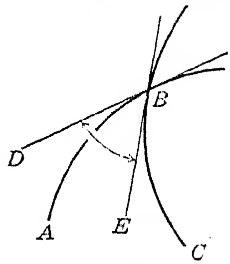



FIG. 221

**200.** THEOREM 58.

> A spherical angle is measured by the arc which it intercepts on that great circle which has the vertex of the given angle as a pole.

*Given:* Sph $\angle APB$. $P$ is a pole of the great circle $c$. Sph $\angle P$ intercepts $\overset{\frown}{AB}$ on $c$.

*Prove:* Sph $\angle APB = \overset{\frown}{AB}$.

1) Draw $OA$, $OB$, $OP$.
2) At $P$ draw $PX$ and $PY$ tangent to $\overset{\frown}{PA}$ and $\overset{\frown}{PB}$, respectively.
3) Show that sph $\angle APB = \angle XPY = \angle AOB = \overset{\frown}{AB}$.

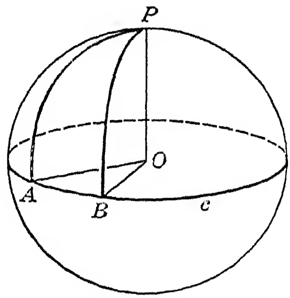



FIG. 222

**201. Spherical Polygons.** If a polyhedral angle (§ 166) has its vertex at the center of a sphere $O$, its edges must pierce the spherical surface at some points $A$, $B$, $C$, $D$, . . . The faces of $O$–$ABCD$ . . . cut the spherical surface in the great circle arcs

$AB$, $BC$, $CD$, . . . The figure $ABCD$ . . . thus formed on the surface of the sphere is called a *spherical polygon* (Fig. 223).

The *vertices* of the spherical polygon are the points $A$, $B$, $C$, $D$, . . .; the *sides* are $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, . . .

$O$–$ABCD$ . . . is the polyhedral angle *corresponding* to spherical polygon $ABCD$ . . .

A spherical polygon is *convex* or *concave* according as its corresponding polyhedral angle is convex or concave (§ 166). Only the *convex* type will be considered here.

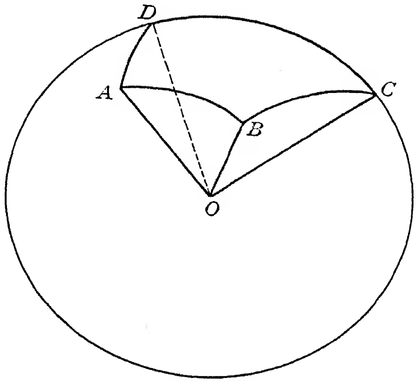



Fig. 223

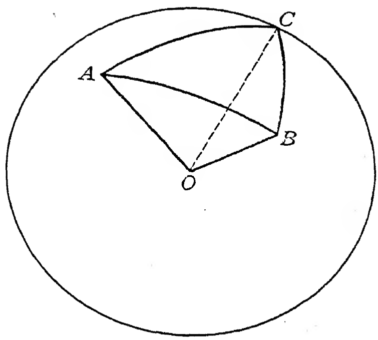



Fig. 224

A spherical polygon is named in accordance with the number of its sides. A *spherical triangle* (Fig. 224) is a spherical polygon with three sides; and of course its corresponding polyhedral angle is a *trihedral angle* of the type presented in Chapter Eleven. A *spherical quadrilateral* has four sides, a spherical *pentagon* has five sides, and so on.

**202.** Two fundamental relations between a spherical polygon and its corresponding polyhedral angle are readily perceived:

A. Each side of the polygon equals (in degrees) some face angle of the polyhedral angle.

B. Each angle of the polygon equals some dihedral angle of the polyhedral angle.

(B is easily proved. For example, in Fig. 223 draw tangents to $\overset{\frown}{AB}$ and $\overset{\frown}{AD}$ from point $A$. The angle formed by these tangents is the measure of sph $\angle DAB$ by § 199; this angle is also the plane angle of dh $\angle D$–$OA$–$B$.)

**203.** In consequence of § 202 together with the material of Chapter Eleven we have the following important facts relating to spherical polygons and spherical triangles. Prove them as exercises.

A. In any spherical polygon each side and each angle is less than 180°.

B. In any spherical polygon the sum of the sides is less than 360° (§ 168).

C. In any spherical triangle the sum of two sides is greater than the third side (§ 167).

D. In any spherical triangle the sum of the angles is greater than 180° and less than 540° (§ 169).

**204. Types of Spherical Triangles.**

A spherical triangle is *isosceles* if two of its sides are equal.

A spherical triangle is *right* if it has at least one right angle.

A spherical triangle is *oblique* if it is not a right triangle.

**205. Right Spherical Triangles.** From § 203–D it is clear that a spherical triangle may have one, two, or even three right angles. If a triangle has two right angles it is called *bi-rectangular;* if it has three, it is *tri-rectangular*.

The following useful facts relating to *right* spherical triangles may be proved as exercises:

A. If two angles of a spherical triangle are right angles, the sides opposite these angles are quadrants; and the third angle equals the third side. (Use §§ 68, 9, Ref. 73, §§ 189, 200.)

B. If two sides of a spherical triangle are quadrants, the angles opposite these sides are right angles; and the angle included by the two given sides is equal to the third side. (Use Ref. 73, §§ 10, 63, 184, 200.)

C. The sides of a tri-rectangular spherical triangle are quadrants.

D. If the sides of a spherical triangle are each quadrants, the triangle is tri-rectangular.

E. In a spherical triangle, if a side and an adjacent angle are each equal to 90°, then the angle opposite the given side and the side opposite the given angle are each equal to 90°.

## EXERCISES

*Group Twenty*

1. Find the spherical distance in nautical miles between two points on the Meridian of Greenwich whose latitudes are respectively 18° 30′ N and 26° 30′ S.

2. Find the spherical distance in nautical miles between two points on the equator whose longitudes are respectively 10° W and 110° E. Find the same distance in ordinary land miles.

3. Find the distance in nautical miles between $A$ (lat. 72° N, long. 23° 10′ W) and $B$ (lat. 18° S, long. 23° 10′ W).

4. Find the distance both in nautical miles and in land miles between $A$ (lat. 30° N, long. 40° E) and $B$ (lat. 30° N, long. 20° W).

5. The sides of a spherical triangle are arcs of 90°, 90°, 40°. Find the number of degrees in each angle of the triangle. If the radius of the sphere is 12″, find the length of each side.

6. The radius of a sphere is 12″. The sides of a triangle on this sphere are $6\pi''$, $6\pi''$, 12″. Find the number of degrees in each side and in each angle of the triangle.

7. Can a spherical triangle have as angles: 80°, 60°, 40°? 90°, 90°, 80°? 120°, 70°, 50°?

8. Can a spherical triangle have as sides: 40°, 70°, 90°? 121°, 118°, 124°? 25°, 70°, 100°?

9. Two sides of a spherical triangle are 115° and 126°. What is the greatest number of degrees possible for the third side?

10. Can a spherical pentagon have as sides: 66°, 105°, 112°, 51°, 88°?

11. Two angles of a spherical triangle are 40° and 130°. Between what two limits is the number of degrees in the third angle?

12. Find the lengths of the sides of a tri-rectangular spherical triangle on a sphere of radius 8″.

13. On a sphere of radius 6 cm. two sides of a triangle are each $3\pi$ cm. and the included angle is 20°. How long is the third side?

14. In a spherical $\triangle ABC$, $A = B = 90°$ and $AB = 4\pi$ in. If the radius of the sphere is 14 in., find the number of degrees in $AC$, $BC$, $C$; find the lengths of sides $AC$ and $BC$.

15. From the center of a sphere of radius 20′ three radii are drawn making an angle of 60° one with another. Find the perimeter (in feet) of the spherical triangle whose vertices are the outer extremities of these radii.

**206. Congruence and Symmetry.** Two spherical polygons are *congruent* if they can be made to coincide. It is evident that two spherical polygons on the same sphere must be congruent if their corresponding polyhedral angles are equal.

Two spherical polygons on the same sphere are *symmetric* if their corresponding polyhedral angles are symmetric. (See § 171.) As in the case of

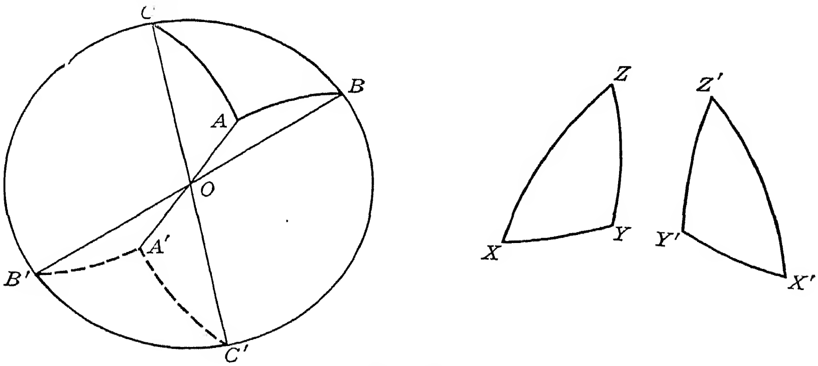



FIG. 225

symmetric polyhedral angles, two symmetric spherical polygons are alike part for part, but they cannot be made to coincide because their respective sets of parts are arranged in opposite orders. (See Fig. 225.)

Show that each of the following is true:

A. Corresponding parts of congruent spherical polygons or of symmetric spherical polygons are equal.

B. Two spherical triangles are congruent if they are mutually equilateral, — corresponding parts being similarly ordered. (See §§ 202, 173.)

C. If two isosceles spherical triangles are symmetric they are also congruent. (See Ex. 8, Group Sixteen.)

D. Two symmetric spherical triangles have equal areas.

(In each triangle choose the point which is the nearer pole of the small circle determined by the vertices. See Fig. 226. Connect this point with each vertex by a great circle arc. Each triangle is now resolved into three isosceles triangles. Show that the three isosceles triangles in the one group are respectively symmetric to the isosceles triangles in the other group. Apply C.)

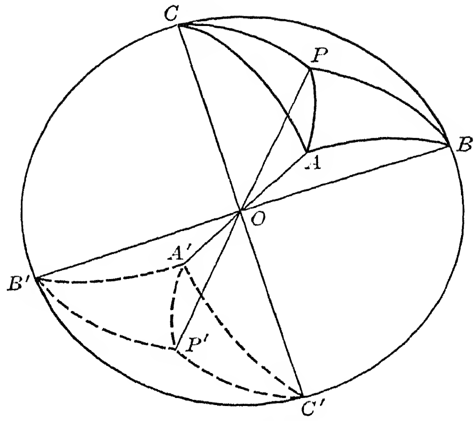



FIG. 226

In a compact course in Solid Geometry §§ 207–209 may be omitted without any serious loss. These sections together with the exercises of Group Twenty-one, however, are of fundamental importance in the work of Chapters Fifteen, Sixteen, Seventeen, Eighteen (Spherical Trigonometry).

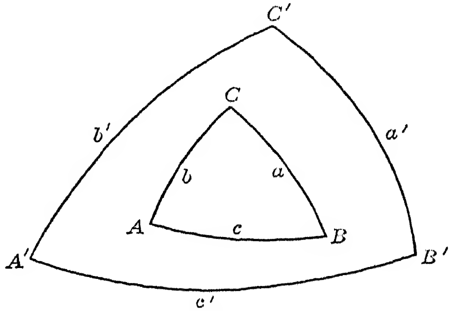



FIG. 227

**207. Polar Triangle.** Let $\triangle ABC$ be any spherical triangle. Draw a second spherical $\triangle A'B'C'$ the sides of which have points $A$, $B$, $C$, respectively, as poles (Fig. 227). Letter $\triangle A'B'C'$ so that $A$ is the pole of $a'$, $B$ is the pole of $b'$, $C$ is the pole of $c'$. The spherical $\triangle A'B'C'$ so constructed is called the *polar triangle* of $\triangle ABC$.

Note that the polar triangle does not necessarily envelop the given triangle, but may lie within it (Fig. 228),

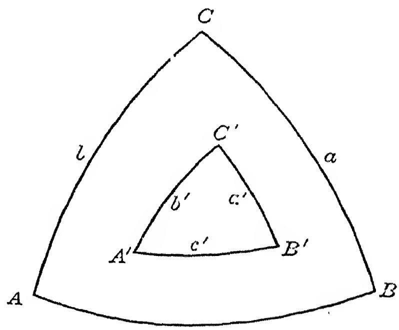



FIG. 228

or it may intersect the given triangle (Fig. 229). The size and position of the polar triangle depend upon the size and shape of the given triangle.

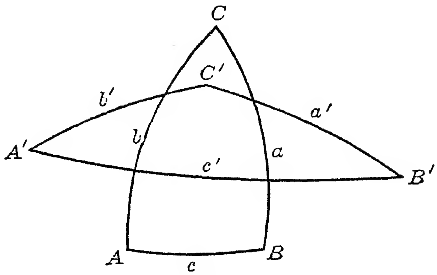



FIG. 229

**208.** THEOREM 59.

If a spherical triangle $XYZ$ is polar to a spherical triangle $ABC$, then reciprocally the triangle $ABC$ is polar to triangle $XYZ$.

*Given:* sph $\triangle XYZ$ polar to sph $\triangle ABC$ (Fig. 230).

*Prove:* sph $\triangle ABC$ is polar to sph $\triangle XYZ$.

We must show that $X$ is a pole of $\overset{\frown}{BC}$, $Y$ is a pole of $\overset{\frown}{CA}$, $Z$ is a pole of $\overset{\frown}{AB}$.

1) Draw great circle arcs $\overset{\frown}{XB}$ and $\overset{\frown}{XC}$.
2) $C$ is a pole of $\overset{\frown}{XY}$. $\therefore$ $\overset{\frown}{XC} = 90°$. $B$ is a pole of $\overset{\frown}{ZX}$. $\therefore$ $\overset{\frown}{XB} = 90°$.
3) $\therefore$ $X$ must be a pole of $\overset{\frown}{BC}$ (§ 189).
4) Similarly, $Y$ and $Z$ are poles of $\overset{\frown}{CA}$ and $\overset{\frown}{AB}$, respectively.
5) $\therefore$ sph $\triangle ABC$ is polar to sph $\triangle XYZ$ (§ 207).

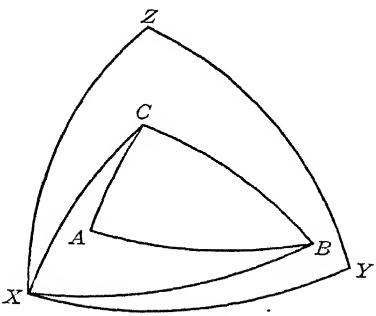



FIG. 230

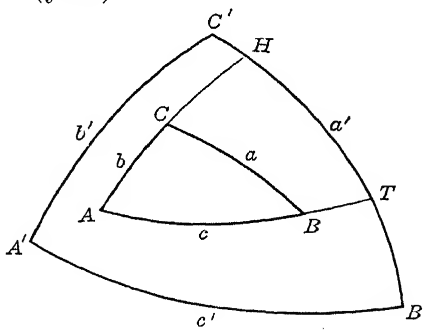



FIG. 231

**209.** THEOREM 60.

If two spherical triangles are polar to each other any angle of one triangle is supplementary to the opposite side of the other triangle.

*Given:* sph $\triangle ABC$ and sph $\triangle A'B'C'$ polar to each other (Fig. 231).

*Prove:* $A + a' = 180°$, $B + b' = 180°$, $C + c' = 180°$; $A' + a = 180°$, $B' + b = 180°$, $C' + c = 180°$.

1) Extend $b$ and $c$ to cut $a'$ at $H$ and $T$, respectively.
2) $A = \widehat{HT}$ (§ 200).
3) $\widehat{C'T} = 90°$ ($C'$ is a pole of $c$)
   $\widehat{HB'} = 90°$ ($B'$ is a pole of $b$)
4) $\therefore\ \widehat{C'T} + \widehat{HB'} = 180°$
5) or $\widehat{HT} + \widehat{C'B'} = 180°$
6) or $A + a' = 180°$

Complete the proof.

## EXERCISES

*Group Twenty-one*

**1.** The sides of a spherical triangle are 30°, 80°, 100°. Find the angles of the polar triangle.

**2.** The angles of a spherical triangle are 40°, 70°, 120°. Find the number of degrees in each side of the polar triangle.

**3.** On a sphere of radius 10″ the angles of a triangle are 60°, 90°, 135°. Find the perimeter (in inches) of the polar triangle.

**4.** Describe the size, shape and position of the triangle which is polar to a given bi-rectangular spherical triangle.

**5.** State the condition or conditions under which: (a) a given spherical triangle and its polar triangle coincide; (b) the polar triangle completely envelops the given triangle; lies within the given triangle.

**6.** Making use of § 209 deduce the fact that the sum of the angles of any spherical triangle is greater than 180° and less than 540°.

**7.** Using §§ 209 and 203–C prove that in any spherical triangle $ABC$: $180° + C > A + B$; $180° + B > A + C$; $180° + A > B + C$.

**8.** In a sph $\triangle ABC$, $b = c$. Prove: $B = C$. (See Fig. 232. Draw median $AM$, i.e., draw $\widehat{AM}$ to bisect $\widehat{BC}$. Use §§ 206–A, B.)

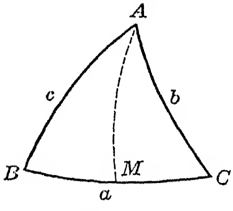



FIG. 232

**9.** On any sphere two spherical triangles are congruent if two sides and the included angle of one respectively equal two sides and the included angle of the other, — all corresponding parts being similarly ordered. Prove. (Draw corresponding trihedral angles.)

**10.** On any sphere two spherical triangles are congruent if two angles and the included side of one respectively equal two angles and the included side of the other, — all corresponding parts being similarly ordered. Prove.

**11.** In a sph $\triangle ABC$, $B = C$. Prove: $b = c$. (Draw the polar triangle. Use § 209 and Ex. 8.)

**12.** On any sphere two spherical triangles are congruent if they are mutually equiangular, — all corresponding parts being similarly ordered. Prove. (Draw the polar triangle for each of the given triangles. Use §§ 209, 206–B.)

**13.** In a sph $\triangle ABC$, $A > B$. Prove: $a > b$. (In the triangle draw $\overset{\frown}{AD}$ to $a$ so that $\angle DAB = \angle B$.)

**14.** In a sph $\triangle ABC$, $a > b$. Prove: $A > B$. (Draw the polar triangle.)

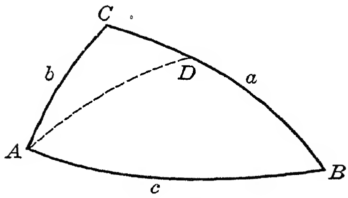



FIG. 233

**15.** Prove that the arcs which bisect the angles of a spherical triangle are concurrent. (Use the corresponding trihedral angle.)

**16.** In Ex. 15 show that if from the point of intersection of the angle bisectors arcs are drawn perpendicular to the sides of the triangle, these arcs are equal.

**17.** Prove that if arcs are drawn to bisect perpendicularly the respective sides of a spherical triangle, these arcs are concurrent at a point whose spherical distances to the three vertices are equal.

**18.** Prove that the medians of a spherical triangle are concurrent.

## OTHER FIGURES ON THE SURFACE OF A SPHERE

**210. Lune** (Fig. 234). A *lune* is a spherical figure bounded by the semi-circumferences of two great circles. In Fig. 234, $ACBDA$ is a lune. The angle $A$ or the angle $B$ is the *angle* of the lune.

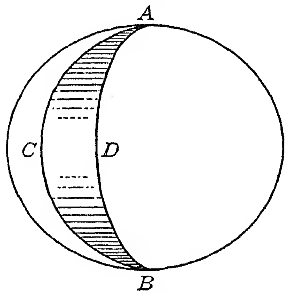



FIG. 234

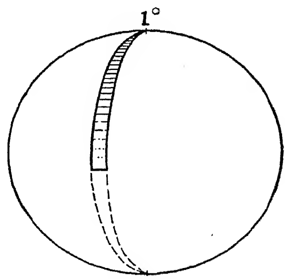



FIG. 235

**211. Spherical Degree** (Fig. 235). A *spherical degree* or *unit triangle* is a spherical triangle the angles of which are, respectively, 90°, 90°, 1°.

It follows at once (§ 205–A) that two of the sides of a spherical degree are quadrants. Hence, a spherical degree is actually one-half of a 1°-lune. Therefore, on any sphere there are exactly 720 spherical degrees.

**212. Dome or Zone of One Base** (Fig. 236). If an arc $AB$ of a great circle on a sphere is rotated about a diameter $AC$, $\widehat{AB}$ generates a *dome* or *zone of one base*.

The point $B$ generates a circle of the sphere; and this circle together with the plane surface bounded by its circumference is the *base* of the zone. The *altitude* is the perpendicular distance ($AD$) between $A$ and the plane of the base. The arc $AB$ is the *generating arc*, and the chord $AB$ is the *generating chord*. Obviously, any circle of a sphere divides the entire surface into two domes.

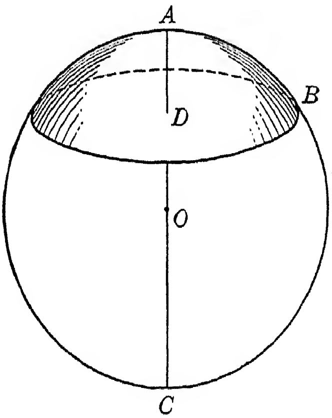



FIG. 236

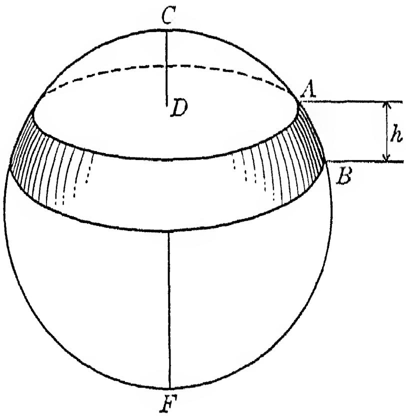



FIG. 237

**213. Zone of Two Bases** (Fig. 237). If an arc $AB$ of a great circle rotates about some fixed diameter which does not meet $\widehat{AB}$, the arc $AB$ generates a *zone of two bases*.

The *bases* are the circles generated by points $A$ and $B$ (together with the plane surfaces bounded by the circumferences of these circles). The *altitude* is the perpendicular distance between the bases. The terms *generating arc* and *generating chord* are used as in the case of a dome.

A zone of two bases is that portion of the surface of a sphere which is included between two parallel circles of the sphere.

The *area* of a zone, either of one base or of two bases is the area of that portion of the spherical surface belonging to that zone.

# *Chapter Fourteen*

## MEASUREMENT OF THE SPHERE

**214.** THEOREM 61.

The area* of a sphere of radius $r$, — area being taken in ordinary square units, — is $4\pi r^2$. $S = 4\pi r^2$.

In order to avoid some confusion, separate the proof into two parts, the first part being a preliminary development necessary to the second.

(a) Fig. 238

*Given:* A line-segment $m$ above a line $XY$. $u$ is a line bisecting $m$ perpendicularly and meeting $XY$ at $D$.
$p$ is the projection of $m$ upon $XY$.

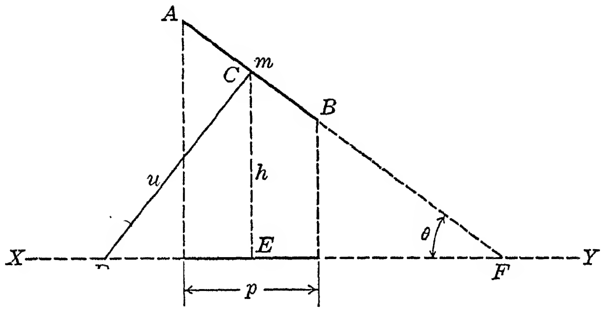



FIG. 238

*Prove:* If $m$ is rotated about $XY$ the area of the surface generated by $m$ is $2\pi up$.

1) From $C$ drop $h \perp XY$, meeting $XY$ at $E$.
Extend $m$ to meet $XY$ at $F$. Let $\angle DFA = \theta$.
Then $\angle DCE = \theta$ (Ref. 41–ii).

2) As $m$ rotates it generates the lateral surface of a frustum of a right circular cone. $h$ is the radius of the mid-section. Therefore the area, $k$, generated by $m$ is $2\pi hm$. (See Ex. 1, Group Fourteen.)

* "Area," of course, means "surface area." See § 181.

3) But $m = p \sec \theta$, and $h = u \cos \theta$.

4) $\therefore \; k = 2\pi(u \cos \theta)(p \sec \theta) = 2\pi up(\cos \theta)(\sec \theta) = 2\pi up(1) = \quad 2\pi up$

(b) Fig. 239

If a quarter-circle of radius $r$ is rotated about a fixed radius $OA$, $\widehat{BA}$ will generate the surface of a hemisphere.

Divide $\widehat{BA}$ into any number of equal arcs. Draw the chords of these arcs. From $O$ draw perpendiculars to these chords. These perpendiculars ($u$) are equal and bisect the chords. $OE$, $EF$, . . . are the projections of these chords upon $OA$.

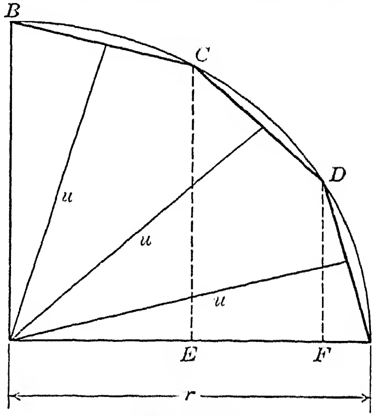



Fig. 239

Now rotate $\widehat{BA}$ about $OA$. From (a) the areas generated by the chords are: $k_1 = 2\pi u \cdot OE$, $k_2 = 2\pi u \cdot EF$, $k_3 = 2\pi u \cdot FA$.

$\therefore \; k$, the area of the entire surface generated by broken line $BCDA$, is

$$2\pi u \cdot OE + 2\pi u \cdot EF + 2\pi u \cdot FA = 2\pi u(OE + EF + FA) = 2\pi ur.$$

Increase the number of divisions on $\widehat{BA}$ indefinitely. $k$ will approach $H$, the area of the hemisphere. Simultaneously, $u$ will approach $r$ as a limit. Since the variables $k$ and $2\pi ur$ are always equal we have: $H = 2\pi rr = 2\pi r^2$. (See Ref. 91.)

$\therefore \; S$, the area of the whole sphere, is $\boxed{4\pi r^2}$

**215. Theorem 62.**

The area of a zone (either of one base or of two bases) is the product of the circumference of a great circle and the altitude of the zone.

$$S = 2\pi rh.$$

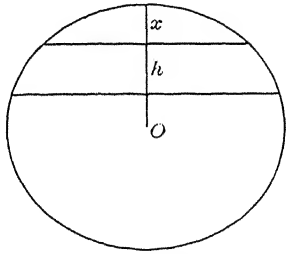



Fig. 240

The proof for the area of a zone of one base is similar to that for § 214.

For a zone of two bases, express the area of the required zone as the difference of the areas of two domes, one having an altitude $x + h$, the other having an altitude $x$ (Fig. 240).

**216.** THEOREM 63.

The area of a lune the angle of which is $n°$ is found by taking $\frac{n}{360}$ the area of the sphere.

For example, if the angle of a lune is 30° it is obvious that 12 of these lunes would comprise the entire spherical surface since $12 \times 30 = 360$. That is, the area of the lune must be $\frac{30}{360}$ or $\frac{1}{12}$ the area of the sphere. This theorem will be assumed without proof.

**217. Spherical Excess** ($E$). The spherical excess of a *spherical triangle* is the number which represents the amount by which the sum of its angles exceeds 180°. Thus, if the sum of the angles of a given spherical triangle is 342°, its spherical excess is the number 162.

The spherical excess of a *spherical polygon* of $n$ sides is the number which represents the amount by which the sum of its angles exceeds $(n-2)180°$. Thus, if the sum of the angles of a spherical hexagon is 850°, the spherical excess is $850 - 4(180)$ or 130.

Spherical excess is to be regarded as a *number*, and is usually represented by the letter $E$.

**218.** THEOREM 64.

The area of a spherical triangle in spherical degrees is $E$. If $r$ is the radius of the sphere the area in ordinary square units is $\frac{E}{180}\pi r^2$.

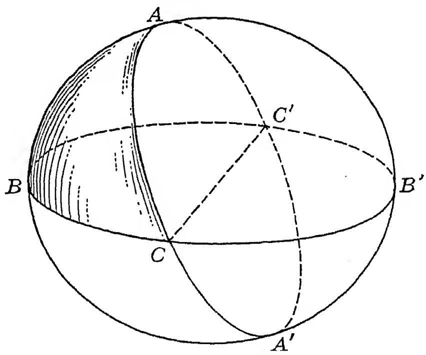


FIG. 241

*Given.* Sph $\triangle ABC$ on sphere of radius $r$.
Nos. of degrees in the angles are $A$, $B$, $C$, respectively.
$E = A + B + C - 180$.
$K$ = area in spherical degrees.
$S$ = area in square units.

*Prove.* (a) $K = E$; (b) $S = \frac{E}{180}\pi r^2$.

(a) Area in spherical degrees.

1) Extend each side of $\triangle ABC$ so as to form three complete great circles. Let the three additional intersections be points $A'$, $B'$, $C'$ (Fig. 241).
2) $L_1$ = lune $ABA'CA = \triangle ABC + \triangle A'BC = K + \triangle A'BC$.
3) But the symmetric triangles $A'BC$ and $AB'C'$ are equal (§ 206–D).
4) $\therefore\ L_1 = K + \triangle AB'C'$.
5) $L_2$ = lune $BCB'AB = \triangle ABC + \triangle AB'C = K + \triangle AB'C$.

6) $L_3 =$ lune $CAC'BC = \triangle ABC + \triangle ABC' = K + \triangle ABC'$.
7) Adding: $L_1 + L_2 + L_3 = 3K + \triangle AB'C + \triangle AB'C' + \triangle ABC'$
$= 2K + (K + \triangle AB'C + \triangle AB'C' + \triangle ABC')$
$= 2K +$ the top hemisphere
$= 2K + 360$ (See § 211).

8) But $L_1 = \frac{A}{360}(720)$, $L_2 = \frac{B}{360}(720)$, $L_3 = \frac{C}{360}(720)$ (§§ 216, 211).

9) or $L_1 = 2A$, $L_2 = 2B$, $L_3 = 2C$.
10) Substituting in (7): $2A + 2B + 2C = 2K + 360$
or $A + B + C = K + 180$.

11) $\therefore$ $K = A + B + C - 180 = E$.

(b) Area in square units.

12) One spherical degree $= \frac{1}{720}$ of the spherical surface $= \frac{1}{720} 4\pi r^2 = \frac{1}{180}\pi r^2$.

13) From (a), $\triangle ABC$ contains $E$ spherical degrees.

14) $\therefore$ $S = E\left(\frac{1}{180}\pi r^2\right) = \frac{E}{180}\pi r^2$

**219. Corollary A** (Th. 64).

The area of a spherical polygon in spherical degrees is $E$. The area in square units is $\frac{E}{180}\pi r^2$.

Draw diagonals from one vertex and thus resolve the given polygon into $(n - 2)$ spherical triangles. Remember that for a spherical polygon $E =$ sum of angles $- (n - 2)180$. Now apply § 218 and add the results.

### EXERCISES

*Group Twenty-two*

**1.** Prove that the areas of two spheres are to each other as the squares of their radii or the squares of their diameters.

**2.** Prove that the area of a dome is equal to the area of a circle the radius of which is the generating chord of the dome.

**3.** On a given sphere any two zones are equal if they have equal altitudes. Prove.

**4.** On a sphere of radius 20″ find the area of a dome of altitude 8″. What fractional part of the entire surface of the sphere does this dome comprise?

**5.** What is the altitude of a dome which comprises one-third of the area of a sphere the radius of which is 8″?

6. The radius of a spherical globe is 12″. A source of light outside the globe is 12″ from the surface. Find the area of the surface which is illuminated.

7. In a problem similar to Ex. 6 what fractional part of the spherical surface is illuminated if the source of light is at a radius' distance from the surface?

8. Theoretically how many miles above the Earth's surface must an observer be in order to see exactly one-twentieth of the Earth's surface?

9. The radius of a sphere is $r''$. A source of light outside the sphere is $n''$ from the surface. Show that the number of square inches of spherical surface illuminated is given by the formula $K = \dfrac{2\pi r^2 n}{r+n}$.

10. On a sphere of radius $n$ cm. the generating arc of a dome is 120°. Find the area of the dome.

11. On a sphere of radius 8″ the radius of one base of a zone of two bases is 8″, and the generating arc is 45°. Find the area of the zone.

12. On a sphere of radius 25″ the radii of the bases of a zone of two bases are respectively 7″ and 24″. Find the area of the zone. Is there more than one answer possible?

13. On a sphere of radius 6″ the plane of the greater base of a zone of two bases is 3″ from the center of the sphere. The generating arc is 30°. Find the area of the zone. Is there more than one answer?

14. On a sphere of radius 26″ the radii of the bases of a zone are each 10″. Find the area of the zone.

15. On the Earth the North Temperate Zone is bounded by the parallels of latitude 30° N and 60° N. Find the number of square (land) miles in the North Temperate Zone.

16. On a sphere of radius $r$ inches the area of a zone of two bases is 348 square inches. The altitude of the zone is 6 inches. Find $r$.

17. Two planes cut a sphere of radius 12″, and are each perpendicular to a diameter $AB$. Find the lengths of the segments intercepted on $AB$ if these planes divide the surface of the sphere into three equal parts.

18. On a sphere of radius 6″ find the area of a 20° lune.

19. On the Earth's surface find the number of square (land) miles contained in one spherical degree.

20. How many square miles of the Earth's surface are bounded by the Meridian of Greenwich and the meridian 70° W?

21. Find the number of degrees in the angle of a lune the area of which is 21 square inches if the area of the sphere is 189 square inches.

22. On a sphere of radius $r$ cm. the area of one spherical degree is 0.5 sq. cm. Find $r$.

23. On a sphere of radius 4″ find the area (in square inches) of a tri-rectangular spherical triangle. What is the area in spherical degrees?

24. What fractional part of the surface of a sphere is contained in a spherical triangle whose angles are 60°, 80°, 150°? What part of the area is contained in a spherical hexagon whose angles are 100°, 120°, 98°, 127°, 175°, 168°?

**25.** On a sphere of radius $8'$ find the area both in square feet and in spherical degrees of a spherical triangle whose angles are $70°$, $70°$, $60°$.

**26.** On a sphere of radius $24''$ find the area (sq. in.) of a spherical triangle whose angles are $110°$, $120°$, $135°$.

**27.** Find each angle of an equiangular spherical triangle which comprises one-twenty-fourth of the area of its sphere.

**28.** The angles of a spherical triangle are proportional to the numbers 3, 4, 5. The area of the triangle is $27\pi$ sq. in. The radius of the sphere is $9''$. Find each angle of the triangle.

**220.** Assumption. If a polyhedron is inscribed in or circumscribed about a sphere, and if the number of faces of the polyhedron is allowed to become infinite, the area of the polyhedron approaches the area of the sphere as a limit, and the volume of the polyhedron approaches the volume of the sphere as a limit.

**221.** Theorem 65.

The volume of a sphere of radius $r$ is $\frac{4}{3}\pi r^3$.

*Given:* Sphere of radius $r$, center $O$. $S$ = area; $V$ = volume.

*Prove:* $V = \frac{4}{3}\pi r^3$.

1) Circumscribe a polyhedron about the sphere. For the polyhedron let $K$ = the area, $W$ = volume.
2) From each vertex of the polyhedron draw a line to $O$. The polyhedron is now seen to be composed of a number of pyramids whose bases are the faces of the polyhedron, and whose altitudes are each equal to the radius of the sphere (§ 194).
3) The volume of any one of these pyramids such as $O$–$ABC$ in Fig. 242 is $\frac{1}{3}r$(area $\triangle ABC$). Hence the sum of the volumes of all these pyramids is $\frac{1}{3}r$(sum of areas of faces). That is, $W = \frac{1}{3}rK$.

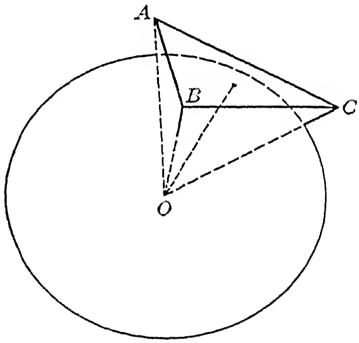



Fig. 242

4) Let the number of faces of the polyhedron become infinite. Then $K \to S$ and $W \to V$ (§ 220). The quantity $\frac{1}{3}r$ remains constant.
5) $\therefore$ the limits $V$ and $\frac{1}{3}rS$ must be equal (Ref. 91).
6) $\therefore$ $V = \frac{1}{3}rS$.
7) But $S = 4\pi r^2$ (§ 214).
8) $\therefore$ $V = \frac{4}{3}\pi r^3$.

*Note:* It is sometimes advantageous to recall that the volume of a sphere is one-third the product of its radius and area. (See step 6 in the proof.)

**222. Spherical Pyramid** (Fig. 243). Let $ABCD$ . . . be any spherical polygon on a sphere $O$. Draw $OA$, $OB$, $OC$, $OD$, . . . The solid $O$–$ABCD$ . . . is a *spherical pyramid.* The *base* is the spherical polygon $ABCD$ . . .; the *vertex* is $O$, the center of the sphere.

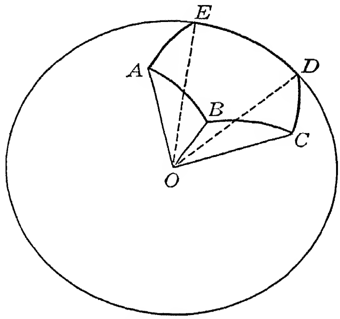



FIG. 243

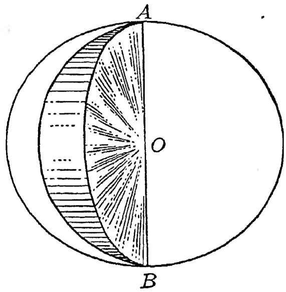



FIG. 244

**223. Spherical Wedge** (Fig. 244). The solid bounded by a lune and the planes of the sides of the lune is a *spherical wedge.* (It is assumed that these planes are not extended beyond the diameter which connects the vertices of the lune.) The *base* of the wedge is the lune itself.

**224. Spherical Sector** (Fig. 245). Let $BOC$ be a sector of a great circle of a sphere $O$. Revolve sector $BOC$ about a fixed diameter $AD$. Sector $BOC$ generates a solid which is called a *spherical sector.* The *base* is the zone traced by the arc $BC$. The *vertex* is $O$, the center of the sphere. The *angle* of the sector is the angle $BOC$.

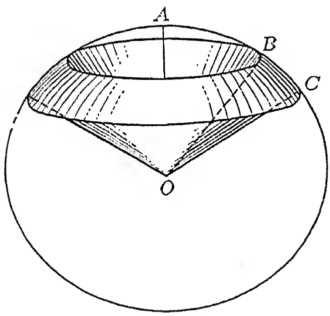



FIG. 245

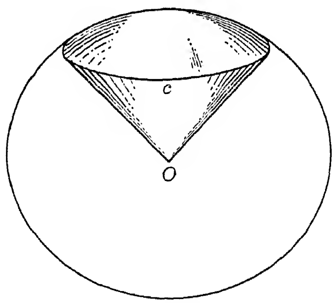



FIG. 246

**225. Spherical Cone** (Fig. 246). Let $c$ be a circle of a sphere $O$. The solid composed of the cone having $O$ as its vertex and circle $c$ as its base together with the solid bounded by the plane of $c$ and the smaller dome having $c$ as base is a *spherical cone.* The *base* is the dome. The *vertex* is $O$.

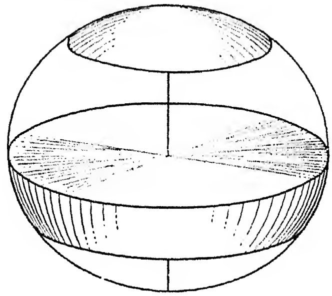


FIG. 247

**226. Spherical Segment** (Fig. 247). A *spherical segment of one base* is the solid bounded by a dome and the base of the dome. The terms "base" and "altitude" are employed as in the case of domes. The *radius* of the segment is the radius of its base.

A *spherical segment of two bases* is the solid bounded by a zone of two bases and the planes of the two bases.

**227.** THEOREM 66.

| |
|---|
| The volume of a spherical $\left\{ \begin{array}{l} \text{pyramid} \\ \text{wedge} \\ \text{sector} \\ \text{cone} \end{array} \right\}$ is one-third the product of the area of its base and the radius of the sphere. $V = \frac{1}{3}br$. |

(a) Volume of Spherical Pyramid (Fig. 248).

*Given:* Sph pyramid $O$–$ABCDE$.
$r$ = radius of sphere.
$V$ = volume.
$b$ = area base.

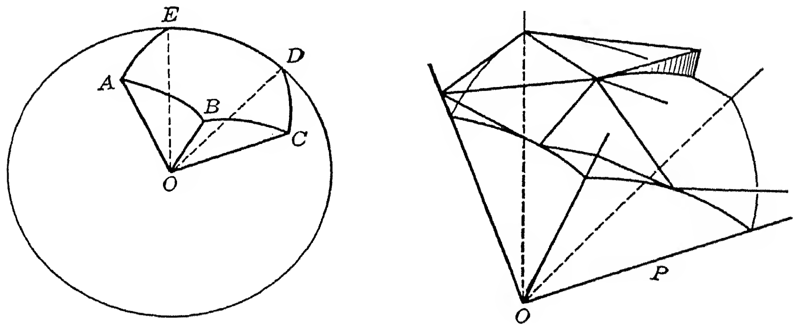



FIG. 248

*Prove:* $V = \frac{1}{3}br$.

Imagine a polyhedron circumscribed about the sphere. Now select merely that portion of this polyhedron which is intercepted by the polyhedral angle $O$–$ABCDE$. Concentrate only upon this portion of the polyhedron; discard the rest of it.

We now have a more or less irregular solid (call it $P$) whose surface is: (i) the faces of polyhedral angle $O\text{–}ABCDE$, and (ii) that portion of the surface of the original polyhedron which has been intercepted by the polyhedral angle. Let $K$ = the area of the surface mentioned in (ii); let $W$ = the volume of $P$.

As in § 221, draw lines to $O$, dividing $P$ into a number of pyramids having $O$ as a common vertex. $r$ is the altitude of each pyramid.

As in § 221, show that $W = \frac{1}{3}rK$.

Let the number of these pyramids become infinite. Then $W \to V$ and $K \to b$.

By Ref. 91 obtain: $V = \frac{1}{3}br$.

(b) Wedge, Sector and Cone

To obtain the volume formulas for each of the other three solids, draw great circle arcs on the base, pass the necessary planes through these arcs and the center of the sphere, and thus resolve the solid into a number of spherical pyramids. Apply part (a), and add the results. Obtain: $V = \frac{1}{3}br$.

**228.** Theorem 67.

In a sphere of radius $r$ the volume of a segment of one base having an altitude $h$ and a radius $a$ is given by either of the two formulas:

$$V = \frac{\pi}{3}h^2(3r - h) \quad \text{or} \quad V = \frac{\pi}{6}h(3a^2 + h^2).$$

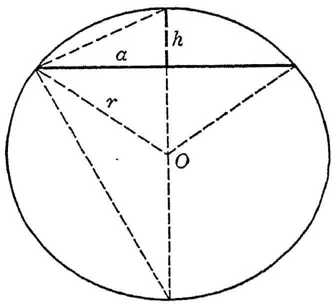



Fig. 249

*Given:* Segment of one base.
$h$ = altitude.
$a$ = radius.
$r$ = radius of sphere.
$V$ = volume.

*Prove:* Formulas stated above.

1) Draw the spherical cone of which this segment is a part.
2) Vol. sph. cone $= \frac{1}{3}(\text{base}) \cdot r$.
3) But area base $= 2\pi rh$.
4) $\therefore$ Vol. sph. cone $= \frac{2}{3}\pi r^2 h$.
5) Altitude of right circular cone in figure is $r - h$. Radius of this cone $= \sqrt{h(2r - h)}$ (Ref. 12–ii.)
6) $\therefore$ Vol. cone $= \dfrac{1}{3}\pi h(2r - h)(r - h) = \dfrac{\pi}{3}h(2r^2 - 3rh + h^2)$

$$= \frac{2}{3}\pi r^2 h - \pi r h^2 + \frac{\pi}{3}h^3.$$

7) $\therefore$ $V$ = vol. sph. cone − vol. cone $\quad \dfrac{\pi}{3}h^2(3r - h)$

8) From the figure, $a^2 = h(2r - h)$ (Ref. 12–ii).

9) Solving for $r$: $r = \dfrac{a^2 + h^2}{2h}$

10) Substitute result in formula obtained in step 7:

$$V = \frac{\pi}{6}h(3a^2 + h^2).$$

*Note:* It is quite possible to derive volume formulas for a spherical segment of two bases. One such formula is $V = \frac{\pi}{6}h(3a^2 + 3b^2 + h^2)$, where $a$ and $b$ are the radii of the bases. In practice, however, it is advisable to obtain the volume of a given segment of two bases by treating it as the difference of the volumes of two segments of one base. (Compare with the method for deriving formula for area of a zone of two bases in § 215.)

## EXERCISES

*Group Twenty-three*

**1.** The volumes of two spheres are to each other as the cubes of their radii or the cubes of their diameters. Prove.

**2.** Find the area and volume of a sphere of radius 6″.

**3.** Assuming that the Earth is a perfect sphere of radius 4000 miles, compute the number of square miles in the Earth's surface. What is the volume?

**4.** The volume of a sphere is $113\frac{1}{7}$ cu. in. Find the area. Let $\pi = \frac{22}{7}$.

**5.** The area of a sphere is 2464 sq. in. Find the volume. Let $\pi = \frac{22}{7}$.

**6.** Find the radius of a sphere for which the number of cubic units of volume is the same as the number of corresponding square units of area.

**7.** The areas of two spheres are respectively 9 and 64. What is the ratio of their radii? their volumes?

**8.** The volumes of two spheres are respectively 27 and 125. What is the ratio of their areas? What is the ratio of their areas if their volumes are respectively $v$ and $w$?

**9.** The radius of one sphere is 3 times that of a second, and the sum of their volumes is $1008\pi$ cu. ft. Find the radius and the area of each sphere.

**10.** The area of one sphere is $k$ times that of a second. What is the ratio of the volume of the first to that of the second?

**11.** The radii of three spheres are proportional to the numbers 1, 2, 3. The sum of their areas is 3584 sq. in. Find the radius and volume of each sphere.

**12.** The angles of a spherical triangle are respectively 75°, 95°, 100°, and the area is $162\pi$ sq. ft. Find the volume of the sphere.

**13.** Find the volume of a spherical wedge the base of which is a 30° lune. The radius of the sphere is 10″.

**14.** The radius of a sphere is 50″. The radius of the base of a dome which forms the base of a spherical cone is 14″. Find the volume of the spherical cone.

**15.** Find the volume of a spherical segment of one base if the altitude is 6″ and the radius of the sphere is 18″.

**16.** The radius of a spherical segment of one base is $\sqrt{21}''$ and the altitude is 3″. Find the volume of the segment and the volume of the sphere.

**17.** The radius of a sphere is 34″. The radius of the greater base of a segment of two bases is 30″. The altitude is 14″. Find the volume of the segment.

**18.** The radii of the bases of a segment of two bases are respectively 40″ and 48″, and the altitude is 16″. Find the volume of the segment. Find the volume of the sphere.

**19.** The base of a spherical sector is a zone of two bases. The altitude is 4″. The radius of the sphere is 20″. Find the volume of the spherical sector.

**20.** The radius of a circle $O$ is 16″. $\overset{\frown}{AB} = 90°$. On arc $AB$ points $C$ and $D$ are taken so that $\overset{\frown}{AC} = \overset{\frown}{CD} = \overset{\frown}{DB}$. $\overset{\frown}{CD}$ is revolved about the radius $OB$. Find the volume of the spherical sector which has $O$ as its vertex and which has as a base the zone generated by $\overset{\frown}{CD}$.

## COLLECTED FORMULAS. SOLID GEOMETRY

### Key to Symbols

| | |
|---|---|
| $b$ = area of base | $h$ = altitude |
| $p$ = perimeter of base | $f$ = slant height |
| $t$ = perimeter of right section | $r$ = radius |
| $e$ = element or lateral edge | $a$ = radius of segment of one base |

| | Lat. Area | Total Area | Volume |
|---|---|---|---|
| Prism | $t \cdot e$ | $2b + t \cdot e$ | |
| Right Prism | $p \cdot h$ | $2b + p \cdot h$ | $b \cdot h$ |
| Cylinder | $t \cdot e$ | $2b + t \cdot e$ | |
| Rt. Circ. Cylinder | $2\pi rh$ | $2\pi r^2 + 2\pi rh$ | $\pi r^2 h$ |
| Cylindric Solid | | | $b \cdot h$ |
| Pyramid | | | $\frac{1}{3}b \cdot h$ |
| Reg. Pyramid | $\frac{1}{2}p \cdot f$ | $b + \frac{1}{2}p \cdot f$ | |
| Frustum Pyramid | | | $\frac{h}{3}(b_1 + b_2 + \sqrt{b_1b_2})$ |
| Frust. Reg. Pyramid | $\frac{1}{2}(p_1 + p_2)f$ | | |
| Cone | | | $\frac{1}{3}b \cdot h$ |
| Rt. Circ. Cone | $\pi r \cdot e$ | $\pi r^2 + \pi re$ | $\frac{1}{3}\pi r^2 h$ |
| Conic Solid | | | $\frac{1}{3}b \cdot h$ |
| Frustum Cone | | | $\frac{h}{3}(b_1 + b_2 + \sqrt{b_1b_2})$ |
| Frust. Rt. Circ. Cone | $\frac{1}{2}(p_1 + p_2)e$ | | |
| Sphere | | $4\pi r^2$ | $\frac{4}{3}\pi r^3$ |
| Zone | | $2\pi rh$ | |
| Lune (of $n°$) | | $\frac{n}{360} 4\pi r^2$ | |
| Spherical Degree | | $\frac{1}{720} 4\pi r^2$ | |
| Spherical △ or Sph. Polygon | | $E$ sph. degrees or $\frac{E}{180} \pi r^2$ sq. units | |
| Sph. Pyramid<br>Sph. Wedge<br>Sph. Sector<br>Sph. Cone | | | $\frac{1}{3}br$ |
| Sph. Segment (1 base) | | | $\frac{\pi}{3} h^2(3r - h)$ or $\frac{\pi}{6} h(3a^2 + h^2)$ |

## MISCELLANEOUS SUPPLEMENTARY EXERCISES

*Group Twenty-four*

1. The volume of a certain solid is 72 cu. in., and its total area is 160 sq. in. Find the total area of a similar solid the volume of which is 9 cu. in.

2. On a sphere of radius 3″ the area of an equiangular spherical triangle is $6\pi$ sq. in. Find the number of degrees in each angle of the triangle.

3. Do Ex. 2 assuming that the spherical triangle instead of being equiangular has angles which are proportional to the numbers 5, 7, 9.

4. The radii of two intersecting spheres are respectively 10″ and 7.5″, and their centers are 12.5″ apart. Find the area of the circle of intersection of the spheres.

5. Find the volume of the cube which can be inscribed in a sphere the area of which is $432\pi$ sq. in.

6. Find the volume of the solid generated by revolving about one of its sides an equilateral triangle one side of which is 8″.

7. The edges of a rectangular solid are respectively $4\sqrt{3}$″, 4″, 6″. Find the length of a diagonal of the solid.

8. Describe the locus of points which are at the same time equidistant from two intersecting planes $M$ and $N$ and at a fixed distance from a fixed point $P$ which is not in either plane.

9. The lateral edges of a triangular truncated prism (§ 87) are each perpendicular to the plane of the base. $DA = 15''$, $EB = 9''$, $FC = 18''$, $AB = 8''$, $BC = 15''$, $AC = 17''$. Find the volume of the solid (Fig. 250).

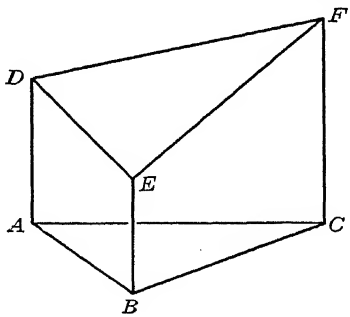



FIG. 250

10. A container is in the form of a hollowed out frustum of a regular square pyramid. $EF = 40''$, $AB = 20''$, altitude $= 24''$. The walls and base of the container are uniformly 6″ thick. Find the amount of material used in making the container. Find the capacity of the container (Fig. 251).

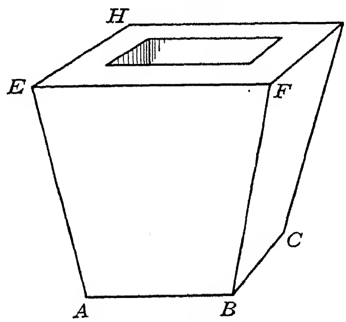



FIG. 251

*Group Twenty-five*

1. An edge of a regular tetrahedron is 20″. Find the area of a mid-section (parallel to one of the faces).

2. Two line-segments $s$ and $t$ are each 12″ long and are everywhere 6″ apart. Describe accurately the locus of points which are at the same time equidistant from $s$ and $t$ and 5″ or less than 5″ from $s$.

3. On a sphere of radius 7″ find the area of a zone of two bases if the altitude of the zone is 2″.

4. Find the number of unit triangles (spherical degrees) in a lune of 40°.

5. Find the number of square inches of area in a spherical degree on a sphere of radius 3″.

6. Find the altitude of a triangular pyramid in which each basal edge is 12″ and each lateral edge is $\sqrt{129}$″.

7. Find the volume of the pyramid of Ex. 6.

8. The basal edges of a right prism are respectively 23.4″, 42.7″, 53.5″. The prism is triangular. The altitude is such that a sphere may be inscribed in the prism. Using logarithms find to four significant figures the volume of the inscribed sphere and the volume of the prism.

9. Each edge of a regular octahedron is $e$. Prove that the volume is $\frac{e^3}{3}\sqrt{2}$.

10. Three line-segments, not necessarily equal, are mutually perpendicular. If $OA$, $OB$, $OC$ are the segments, and if $OP$ is perpendicular to the plane of $A$, $B$, $C$, prove that $P$ is the orthocenter of the triangle $ABC$.

*Group Twenty-six*

1. Two sides of a spherical triangle are each 90° and the included angle is 40°. Find the area of the triangle (sq. in.) if the volume of the sphere is $288\pi$ cu. in.

2. In a sphere of radius 12″ find the volume of a spherical wedge the base of which is a 15° lune.

3. Find the volume of a frustum of a right circular cone if the altitude of the frustum is 12″ and the radii of the bases are respectively 10″ and 19″.

4. Find the lateral area of the frustum of Ex. 3.

5. Find the total area of the circular cone which can be inscribed in a regular tetrahedron one edge of which is 6″.

6. The base of a certain cone is two-thirds the base of a certain cylinder; and the volume of the cone is four-fifths the volume of the cylinder. Compare the altitudes of the two solids.

7. $\triangle ABC$ has $a = 7''$, $b = 5''$, $c = 4''$. Side $a$ is parallel to a plane $M$, and the plane of $\triangle ABC$ makes an angle of 60° with $M$. Find the area of the projection of $\triangle ABC$ upon $M$.

8. Between what limits must the sum of the dihedral angles of a trihedral angle always lie? Prove your answer.

9. A cone is completely filled with ice cream and enough more ice cream is added to the top so that the whole assumes the shape of a spherical cone. The spherical radius is 5″ and the altitude of the dome which forms the base is 0.2″. Find the total amount of ice cream. (Neglect the thickness of the wall of the ice cream cone itself.)

10. Find the number of cubic inches of metal in the wall of a spherical metal shell if the wall is uniformly 1″ thick and if the inner radius is 15″.

### *Group Twenty-seven*

1. In a sphere of radius 6″ the base of a spherical pyramid is a spherical pentagon the angles of which are 80°, 100°, 104°, 136°, 150°. Find the volume of the spherical pyramid.

2. The slant height of a right circular cone $C$ is 10 times the radius of its base. The total area is the same as that of a certain sphere $T$. Find the ratio of the volume of $C$ to that of $T$.

3. The total area of a cube is 72 sq. in. Find the volumes of the inscribed and circumscribed spheres.

4. In a pyramid $V\text{-}ABC$, $VA = 8''$, $VB = 6''$, $VC = 5''$. A plane $M$ cuts $VA$ at $D$, $VB$ at $E$, $VC$ at $F$. $VD = 4''$, $VE = 5''$, $VF = 2''$. If the volume of $V\text{-}ABC$ is 48 cu. in., find the volume of $V\text{-}DEF$.

5. The radius of a sphere is 10″. A plane $M$ cuts the sphere and bisects a radius perpendicularly. Find the polar distance (in inches) of the circle of the sphere determined by plane $M$.

6. Two lines $s$ and $t$ intersect at $A$. Describe accurately the locus of points which are at the same time equidistant from $s$ and $t$ and 5″ from $A$.

7. All the edges of a regular hexagonal prism are equal. The volume is $12\sqrt{3}$ cu. in. Find the altitude.

8. Diameter $AB$ of a certain sphere is 6″. Two planes each perpendicular to $AB$ cut the sphere so as to divide the volume into three equal parts. Find the lengths of the segments intercepted on $AB$ by these planes.

9. The base of a cone is a circle of radius 6″. $A$ and $B$, two points on the circumference of this circle, determine an arc of 90°. $V$ is the vertex of the cone; the altitude is 15″. A plane containing $V$, $A$, $B$ is drawn. Find the volumes of the two solids into which this plane divides the given cone.

10. By definition, the three conditions for a regular polyhedron are: (a) faces are regular polygons, (b) faces are congruent polygons, (c) polyhedral angles are equal.

By giving an appropriate illustration in each case, show that if *any two* of the above conditions hold true in a given polyhedron, then the *third* condition does *not* necessarily hold true, and hence the given polyhedron is not necessarily regular.

### *Group Twenty-eight*

1. Two angles of a spherical triangle are respectively 80°, 92°. The area of the triangle is $4\pi$ sq. in., and the radius of the sphere is 6″. Find the third angle of the triangle.

2. Each element of a circular cone is 15″, and the lateral area is $105\pi$ sq. in. Find the area of an axial section.

**3.** Find the complete length of the locus of points which are 6″ from each of two fixed points which are 8″ apart.

**4.** Find the number of degrees in each dihedral angle of a regular tetrahedron.

**5.** The basal edges of a regular triangular pyramid are each 6″. The altitude makes an angle of 30° with each lateral edge. Find the length of each lateral edge.

**6.** A sphere of radius $x$ is inscribed in a right circular cone of slant height $f$ and radius $r$. Show that
$$x = r\sqrt{\frac{f - r}{f + r}}.$$

**7.** Find the radius of the sphere which can be circumscribed about a regular octahedron each edge of which is 8″. Find, also, the radius of the inscribed sphere.

**8.** In a sphere of radius 10″ find the volume of a spherical sector the base of which is a zone of two bases having an altitude of 3″.

**9.** The volume of a circular cone is 81 cu. in. A plane parallel to the base cuts the cone and determines a frustum of altitude 7″ and volume 57 cu. in. Find the altitude of the original cone.

**10.** $O$ is the geometric center of a regular icosahedron and $ABC$ is one face. Find the number of degrees in each of the equal dihedral angles $OA$, $OB$, $OC$ of the trihedral angle $O$–$ABC$. (Circumscribe a sphere about the icosahedron. Points $A$, $B$, $C$ determine a spherical triangle $ABC$ upon this sphere. What is the relation between dh $\angle OA$ and $\angle A$ of the sph $\triangle ABC$? What fractional part of the surface of the sphere is contained in sph $\triangle ABC$?)

*Group Twenty-nine*

**1.** A solid metal pyramid is cut into two parts by a plane parallel to the base and midway between vertex and base. Find the ratio of the weights of the two resulting parts of the solid.

**2.** Point $P$ is 1″ from a plane $M$. Describe accurately the locus of points which are at the same time 10″ from $P$ and 7″ from $M$.

**3.** Find the area of a sphere circumscribed about a rectangular solid the edges of which are respectively 8″, 8″, 4″.

**4.** How far from the surface of a sphere of radius 4″ must a source of light be placed so that exactly one-sixth of the surface of the sphere will be illuminated?

**5.** The sides of a spherical triangle are 70°, 60°, 94°. Find the area (in square inches) of the polar triangle if the radius of the sphere is 20″ (§§ 207–209).

**6.** The surface of one sphere is twice that of a second, and the sum of the volumes is 14 cu. in. Find the volume of each.

**7.** Describe the locus of points which are equidistant from three planes $M$, $N$, $S$ if $M$ is parallel to $N$, and if $S$ intersects $M$ and $N$.

**8.** Prove that the volume of any triangular truncated prism (§ 87) whose lateral edges are each perpendicular to the plane of the base is the product of one-third the sum of the lateral edges and the area of the base.

**9.** A right circular cone with open base is to be made from a single piece of heavy paper. The altitude of the cone is to be 20″ and the diameter of the open base is to be 30″.

Describe the shape and give accurately the dimensions of the paper pattern from which the cone can be made. (Disregard any over-lap for the seam.)

**10.** $ABC$ is a tri-rectangular spherical triangle on a sphere of radius $3\sqrt{2}''$. Three arcs meeting at a point $D$ within the triangle are known to bisect the three angles. Find the lengths (inches) of each of the arcs $DA$, $DB$, $DC$.

*Group Thirty*

**1.** Each side of an equilateral $\triangle XYZ$ is $12''$. Describe accurately the locus of points which are at the same time equidistant from $X$ and $Y$ and $10''$ or less than $10''$ from $Z$.

**2.** Four points $A$, $B$, $C$, $D$ do not lie all in one plane and no three are collinear. Explain how to construct the sphere the surface of which shall contain the four given points.

**3.** One angle of a spherical triangle is $115°$ and the sides including this angle are equal. Find each of the other two angles if the triangle contains 99 spherical degrees. (See Ex. 8, Group Twenty-one.)

**4.** The number of square feet in the total area of a certain cylinder of revolution is known to be 8 times the number of cubic feet in its volume. The radius is 3 times the altitude. Find the volume.

**5.** Describe the locus of points which lie on a given sphere and which are: (a) equidistant from two fixed points on the sphere; (b) equidistant from the center of the sphere and a fixed point on the sphere; (c) equidistant from two fixed radii.

**6.** The radius of a sphere is $10''$, and $P$ is a fixed point which is $20''$ from the center. From $P$ a line is drawn tangent to the sphere at a point $A$. If $PA$ rotates about $PO$ as an axis find: (a) the length of the curve traced by point $A$; (b) the area of the surface generated by the line-segment $PA$.

**7.** The altitude of a cone of revolution is $6''$ and the diameter is $4\sqrt{3}''$. A sphere is inscribed in the cone and is tangent to the conic surface along a circle $k$. Find the circumference of $k$.

**8.** $ABCD\text{–}EFGH$ is a right prism. $AE$ and each of the other lateral edges is $12''$. $AB = 16''$, $BC = 20''$, $DA = 12''$. $\angle DAB = \angle ABC = 90°$. (a) Find to the nearest tenth of a square inch the total area of the prism. (b) Draw AF and AG and find the number of degrees in $\angle GAF$. (c) Find the volume of the pyramid $A\text{–}BCGF$.

**9.** A sphere is circumscribed about a regular hexagonal prism. The radius of the sphere is twice the altitude of the prism. Find the ratio of the volume of the sphere to that of the prism.

**10.** $M\text{–}x\text{–}N$ is an acute dihedral angle which is greater than $30°$. $M$ is taken horizontal, and $N$ meets $M$ from above in a line $x$. Point $P$ is above $N$ and is $4''$ both from $x$ and $N$. Show how to construct through $P$ a straight line $t$ which shall be parallel to $M$ and which shall intersect $N$ at an angle of $30°$.

*Group Thirty-one*

**1.** Two equal right circular cylinders are placed so that the axis of one exactly coincides with an element of the other. If the altitude and radius of each are respectively $6''$ and $3''$, find the volume which is common to the two cylinders.

**2.** The area of the base of an oblique prism is 40 sq. in. Each lateral edge is $7''$ and makes an angle of $45°$ with the plane of the base. Find the area of a right section and the volume.

3. In a sphere of radius 30″ the altitude of a segment of one base is 6″. Find the volume of the segment.

4. The radius of an equilateral cone is 18″. Find the volume and area of the inscribed sphere.

5. In Ex. 4 find the total area of the inscribed regular triangular pyramid.

6. $ABC$–$DEF$ is a frustum of a triangular pyramid. $x$ = area of $\triangle ABC$, $y$ = area of $\triangle DEF$, $h$ = altitude. Draw a plane through $E$, $A$, $C$ and a plane through $E$, $A$, $F$. Prove that the volume of pyramid $E$–$ACF$ is $\frac{h}{3}\sqrt{xy}$ (Fig 252).

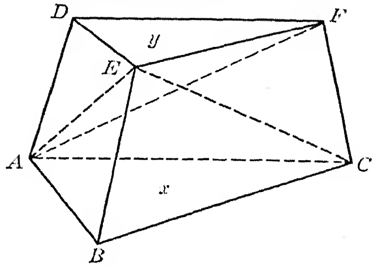



FIG. 252

7. The altitude of a right circular cone is 12″ and the radius is 9″. A sphere of radius 10″ has its center at the vertex of the cone. Find the volume of that portion of the cone which is outside the sphere.

8. $a$, $b$, $c$ are three skew lines. Explain how to construct a line $l$ which will intersect the three given lines.

9. $x$ and $y$ are two lines in space, and $P$ is a point not on either line. $x$ is parallel to the plane of $P$ and $y$; $y$ is parallel to the plane of $P$ and $x$. Prove that $x$ is parallel to $y$.

10. $ABCD$ is a square piece of tin 12″ on a side. $X$ and $Y$ are respectively the midpoints of $BC$ and $CD$. Draw $AX$, $XY$, $YA$. Fold the tin along these three lines and thus form a triangular pyramid the base of which is $\triangle AXY$. Points $B$, $C$, $D$ will coincide with one another at some point $V$. Find the volume of the pyramid $V$–$AXY$.

*Group Thirty-two*

1. Spherical triangles $ABC$ and $A'B'C'$ are polar to each other. $O$–$ABC$ and $O$–$A'B'C'$ are the corresponding trihedral angles. Interpret in terms of face angles and dihedral angles of $O$–$ABC$ and $O$–$A'B'C'$ the relations which exist between the angles of $\triangle ABC$ and the sides of $\triangle A'B'C'$ (§§ 207–209).

2. Can a spherical triangle have 130°, 70°, 55° as sides? 150°, 78°, 100°? 130°, 120°, 115°? Can a spherical triangle have 90°, 80°, 40° as angles? 50°, 40°, 48°? Justify your answer in each case.

3. $P$ is the common pole to two small circles $c$ and $d$. The polar distance of $c$ is 30° and that of $d$ is 60°. If the radius of the sphere is 10″ find the area of the zone of two bases determined by $c$ and $d$.

4. $V$–$ABC$ is a regular tetrahedron each edge of which is $e$. Find the length of the line-segment which is perpendicular to $AB$ and $VC$.

5. A right circular cone is inscribed in a sphere. If the altitude of the cone is three-fourths the diameter of the sphere, what is the ratio of the volume of the cone to the volume of the sphere?

6. $A$ is a fixed point on a fixed line $x$. Line-segment $AB$ is 6″ and $AB \perp x$. $CD$ is a line-segment bisected by point $B$ and is coplanar with $AB$ and $x$. $CD = 3''$. If $CD$ makes a constant angle with $AB$, and if $AB$ is rotated through 360° about $x$, find the areas of the surfaces generated by $AB$ and $CD$, respectively.

7. Prove that the volume of a right prism with triangular bases is one-half the product of the area of one lateral face and the length of the perpendicular drawn to the plane of that face from the opposite lateral edge.

8. A right circular cylinder of radius 12″ is partially filled with water. A spherical metal ball of radius 12″ is dropped into the water. After the ball rests on the bottom of the cylinder the level of the water is found to be 24″ above the base. How many cubic inches of water are there in the cylinder, and what was the depth of the water before the ball was dropped in?

9. $M$-$AB$-$N$ is a dihedral angle. $D$ lies on $AB$ (between $A$ and $B$). $\angle CDE$ is the plane angle of the dihedral, with $C$ lying in $M$ and $E$ lying in $N$. In $M$ line $DX$ is drawn between $DC$ and $DB$; in $N$ line $DY$ is drawn between $DE$ and $DB$. Prove that $\angle XDY$ is less than $\angle CDE$.

10. Prove that the arcs which bisect the sides of a spherical triangle perpendicularly are concurrent.

*Group Thirty-three*

1. If corners are cut from a cube by planes which pass through the mid-points of the edges, what fractional part of the original volume remains?

2. $b_1$ and $b_2$ are the areas of the lower and upper bases, respectively, of a frustum of a pyramid. $h$ is the altitude. Extend the lateral edges of the frustum to form the complete pyramid of which the given frustum is a part. Let $x$ be the altitude of the complete pyramid. Prove:

$$x = \frac{h(b_1 + \sqrt{b_1 b_2})}{b_1 - b_2}.$$

3. The altitude of a spherical segment of two bases is $h$. The radii of the two bases are respectively $a$ and $b$. Prove that the volume of the segment is $\frac{\pi}{6}h(3a^2 + 3b^2 + h^2)$.

4. A hemisphere is tangent to a horizontal plane $M$, and the plane of its circular base is above $M$ and parallel to $M$. A right circular cone and a right circular cylinder, each with a radius and altitude equal to the radius of the hemisphere, rest upon the same plane. A plane $S$ parallel to $M$ cuts these three solids thus forming three circular sections. Prove that the circular section of the cylinder equals the sum of the circular sections of the cone and hemisphere.

5. $ABCD$-$XYZW$ is any parallelepiped, and $P$ is any point in space. Prove that the planes $PAX$, $PBY$, $PCZ$, $PDW$ have one line in common.

6. Four equal spheres each of radius 2″ are placed so that each sphere is tangent to each of the other three. How long is the edge of the smallest regular tetrahedron which will exactly enclose these spheres?

7. A line-segment $CD$ is 12″ long and is a diameter of a circle with center $O$. A chord $AB$ bisects radius $OC$ perpendicularly at a point $M$. With $D$ as a center and $DA$ as a radius the minor arc $AB$ is drawn. Minor arc $AB$ together with the arc $ACB$ thus form a cres-

cent. Rotate the figure about $CD$ and compute the volume and total area of the solid generated by the crescent.

**8.** The diagonals of a rhombus $R$ are respectively $d_1$ and $d_2$. $R$ is first revolved about $d_1$ as an axis, generating a solid of volume $V_1$. $R$ is then revolved about $d_2$ as an axis generating a solid of volume $V_2$. Prove:

$$\frac{V_1}{V_2} = \frac{d_2}{d_1}.$$

**9.** The edges of a rectangular solid which meet at a common vertex $P$ are $PA$, $PB$, $PC$, having as respective lengths $a$, $b$, $c$. Draw $AB$, $BC$, $CA$. Prove that the area of $\triangle ABC$ is

$$\tfrac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}.$$

**10.** Three non-intersecting circles with centers $A$, $B$, $C$ lie in a plane $M$. The circles are unequal and their centers are non-collinear. A pair of common external tangents is drawn to each of the three pairs of circles. The pair of common external tangents to circles $A$ and $B$ meet at a point $Z$; the external tangents to $B$ and $C$ meet at $X$; the tangents to $A$ and $C$ meet at $Y$. Prove that the points $X$, $Y$, $Z$ are collinear.

(Hint: Three points will be collinear if they can be shown to lie on the intersection of two planes. At $A$ draw a line-segment $AD \perp M$. Draw $DY$ and $DZ$. Show that lines perpendicular to $M$ at $B$ and $C$ must cut $DZ$ and $DY$ at some points $E$ and $F$, respectively. Show that points $E$, $F$, $X$ are collinear, and hence that $DZ$, $DY$, $EX$ lie in one plane $S$.)

# *Chapter Fifteen*

## SPHERICAL TRIGONOMETRY: THE SPHERICAL RIGHT TRIANGLE

In Chapters Twelve through Fourteen relating to the geometry of the sphere a portion of the work was devoted to an elementary study of spherical angles and spherical triangles. In the work to follow we shall utilize much of this earlier material to develop a study of spherical triangles from a trigonometric standpoint. This particular phase of the study of spherical triangles is known as Spherical Trigonometry. It will be seen presently that Spherical Trigonometry bears the same relation to the early study of spherical triangles that Plane Trigonometry does to the treatment of ordinary triangles in Plane Geometry.

Spherical Trigonometry is concerned with solving for certain parts (angles or sides) of a spherical triangle when certain other parts are known. For this purpose a series of formulas will be developed. In these formulas it is to be understood that *sides* as well as *angles* are measured in *degrees* instead of in linear units. Only the ordinary type of spherical triangle will be considered, namely, one in which each side is an arc of a great circle, and one in which each side and each angle is less than 180°.

This type of study necessarily presupposes not only familiarity with Chapters Twelve, Thirteen, Fourteen, but also a knowledge of Plane Trigonometry and facility in the use of standard trigonometric and logarithmic tables. At the end of Chapter Eighteen are listed the standard formulas encountered in Plane Trigonometry. Reference to any of these will be made by prefixing a "T" to the number of the formula cited. Thus, "T 5" will mean "formula 5" of this reference list.

### SECTION ONE. PRELIMINARY

The following facts already presented in the earlier work are fundamental to the development of Spherical Trigonometry. They are repeated here for convenient reference.

**229.**

In any spherical triangle $ABC$, if $a = b$, then $A = B$, and conversely.
In any spherical triangle $ABC$, if $a > b$, then $A > B$, and conversely.

**230.**

> If two spherical triangles $ABC$ and $A'B'C'$ are polar to each other, any side of either triangle is supplementary to the opposite angle of the other triangle.

**231.**

> In a spherical triangle $ABC$, if $a = b = 90°$, then $A = B = 90°$, and $c = C$. Conversely, if $A = B = 90°$, then $a = b = 90°$, and $C = c$.

**232.**

> In a spherical triangle $ABC$, if $a = b = c = 90°$, then $A = B = C = 90°$, and conversely.

**233.**

> In a spherical triangle $ABC$, if $a = B = 90°$, then $A = b = 90°$.

**234. Species.** Any two parts of a spherical triangle are said to be of the *same species* if the two parts are each acute (less than 90°), or if the two parts are each obtuse (greater than 90° and less than 180°). Two parts are said to be of *different species* if one is acute and the other is obtuse.

**235.**

In a spherical triangle $ABC$ in which $C = 90°$ (and neither of the other angles is 90°), $A$ and $a$ are of the same species, $B$ and $b$ are of the same species.

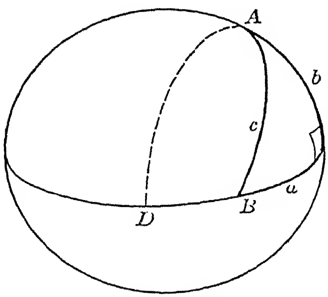



Fig. 253

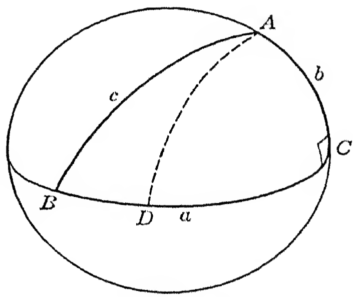



Fig. 254

Part I (Fig. 253).

*Given:* $C = 90°$, $A < 90°$. *Prove:* $a < 90°$.

Conversely, *given:* $a < 90°$. *Prove:* $A < 90°$.

1) Extend $\overset{\frown}{CB}$. Draw $\overset{\frown}{AD}$ making sph $\angle CAD = 90°$.
2) Then $\overset{\frown}{DC} = 90°$ (§ 231).
3) Since $\overset{\frown}{AB}$ falls between $\overset{\frown}{AD}$ and $\overset{\frown}{AC}$, $\overset{\frown}{BC}$ must be less than $90°$.
4) $\therefore a < 90°$.
5) Conversely, if $a < 90°$, extend $\overset{\frown}{CB}$ to a point $D$ to make $\overset{\frown}{CBD} = 90°$. Then $\angle CAD = 90°$ (§ 233). Proceed as before.

PART II (Fig. 254).

*Given:* $C = 90°$, $A > 90°$. *Prove:* $a > 90°$.
Conversely, *given:* $a > 90°$. *Prove:* $A > 90°$.

6) As before, draw $\overset{\frown}{AD}$ making sph $\angle CAD = 90°$.
7) Show that $\overset{\frown}{DC} = 90°$, and hence that $a > 90°$.
8) Conversely, if $a > 90°$ choose $D$ on $\overset{\frown}{BC}$ so that $\overset{\frown}{DC} = 90°$. Show that $\angle CAD = 90°$ and hence that $\angle CAB > 90°$.

Therefore, from Parts I and II: $A$ and $a$ must be of the same species. Similarly, $B$ and $b$ must be of the same species.

**236.**

In a spherical triangle $ABC$ in which $C = 90°$ (and neither of the other angles is $90°$), if $a$ and $b$ are of the same species, then $c$ is acute; if $a$ and $b$ are of different species, then $c$ is obtuse.

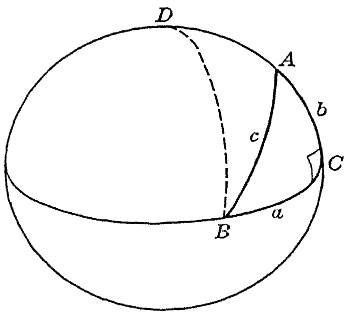



FIG. 255

PART I (Fig. 255).

*Given:* $C = 90°$, $a < 90°$, $b < 90°$. *Prove:* $c < 90°$.

1) Extend $\overset{\frown}{CA}$ to $D$ making $\overset{\frown}{CAD} = 90°$. Draw $\overset{\frown}{DB}$.
2) $\therefore \overset{\frown}{DB} = 90°$ (§ 233).
3) $\therefore \angle BDC = a$ (§ 231). Hence, $\angle BDC < 90°$.
4) Moreover, $\angle BAC < 90°$ (§ 235).
5) $\therefore \angle BAD > 90°$.
6) $\therefore$ in $\triangle DAB$: $\angle BDA < \angle BAD$

7) $\therefore\ \overset{\frown}{AB} < \overset{\frown}{DB}$ (§ 229).
8) $\therefore\ c < 90°$.

PART II (Fig. 256).

*Given:* $C = 90°$, $a > 90°$, $b > 90°$. *Prove:* $c < 90°$.

9) On $\overset{\frown}{CA}$ take $\overset{\frown}{CD} = 90°$. Draw $\overset{\frown}{DB}$. Proceed as in Part I.

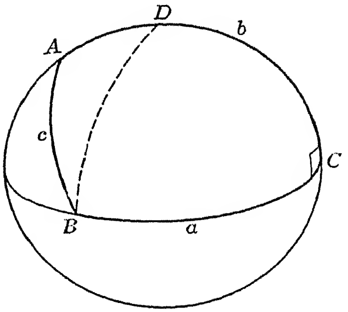



FIG. 256

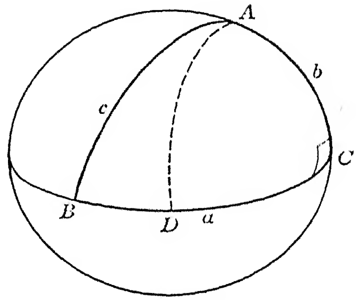



FIG. 257

PART III (Fig. 257).

*Given:* $C = 90°$, $a > 90°$, $b < 90°$. *Prove:* $c > 90°$.

10) On $\overset{\frown}{CB}$ take $\overset{\frown}{CD} = 90°$. Draw $\overset{\frown}{DA}$. Proceed as before.
Thus, Parts I, II, III establish the theorem.

## SECTION TWO. THE SPHERICAL RIGHT TRIANGLE HAVING ONLY ONE RIGHT ANGLE

If a spherical triangle has three right angles, the sides opposite these angles are quadrants (§ 232). If a spherical triangle has two right angles, the sides opposite these angles are quadrants, and the third side and third angle have the same measurement (§ 231). Therefore, the solution of the tri-rectangular triangle or the solution of the bi-rectangular triangle presents no great problem. Hence we need to concentrate upon the right triangle having only *one right angle*.

In the following section (§ 237) we shall develop ten formulas for a spherical right triangle $ABC$ in which only one angle, $C$, equals 90°. For simplicity's sake we shall first develop these formulas with respect to a triangle in which no part is obtuse. But in § 238 which follows, we shall establish the validity of these formulas for *any* spherical triangle $ABC$ in which one angle, $C$, is 90°.

**237. The Ten Formulas** (Fig. 258). Let $O$ be the center of the sphere. Draw $OA$, $OB$, $OC$. Then $\angle s$ $BOC$, $COA$, $BOA$ are equal respectively to $a$, $b$, $c$ of sph $\triangle ABC$.

Draw $BD \perp OC$. In the plane of $AOC$ draw $DE \perp OA$. Draw $BE$. Then $\angle BED = A$ of sph $\triangle ABC$.

Denote $ED$, $BD$, $BE$ by $x$, $y$, $z$, respectively.

Denote $OE$, $OD$, $OB$ by $m$, $h$, $r$, respectively.

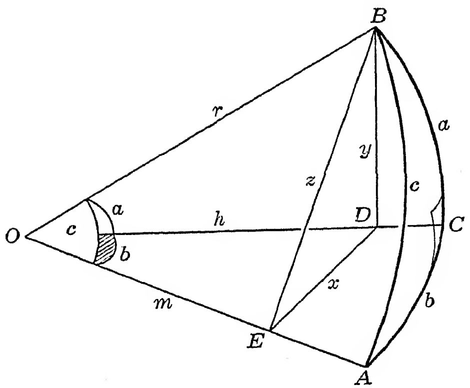



FIG. 258

1) In $\triangle OED$: $m = h \cos b$.

2) In $\triangle OEB$: $m = r \cos c$; in $\triangle ODB$: $h = r \cos a$.

3) Substituting in 1): $r \cos c = r \cos a \cos b$

or

$$\text{①} \qquad \cos c = \cos a \cos b$$

4) In $\triangle EDB$: $\cos A = \frac{x}{z}$.

5) In $\triangle OED$: $x = m \tan b$; in $\triangle OEB$: $z = m \tan c$.

6) Substituting in 4): $\cos A = \dfrac{m \tan b}{m \tan c}$

or

$$\text{②} \qquad \cos A = \tan b \cot c$$

7) From a figure similar to Fig. 258 with $A$ and $B$ interchanged we may obtain:

$$\text{③} \qquad \cos B = \tan a \cot c$$

8) In $\triangle ODB$: $\sin a = \frac{y}{r}$.

9) In $\triangle EDB$: $y = z \sin A$; in $\triangle OEB$: $r = z \csc c$.

10) Substituting in 8): $\sin a = \dfrac{z \sin A}{z \csc c}$

or

$$\text{④} \qquad \boxed{\sin a = \sin A \sin c}$$

11) Similarly:

$$\text{⑤} \qquad \boxed{\sin b = \sin B \sin c}$$

12) In $\triangle EDB$: $x = y \cot A$.

13) In $\triangle OED$: $x = h \sin b$; in $\triangle ODB$: $y = h \tan a$.

14) Substituting in 12): $h \sin b = h \tan a \cot A$

or

$$\sin b = \cot A \tan a$$

15) Similarly: (7) $\sin a = \cot B \tan b$

16) In (6) set $\tan a = \dfrac{\sin a}{\cos a}$. $\therefore \sin b = \cot A \left(\dfrac{\sin a}{\cos a}\right)$.

17) $\therefore \cos a = \cot A \sin a \csc b$.

18) From (7): $\cos b = \cot B \sin b \csc a$, by similar steps.

19) Substitute 17) and 18) in (1):

$\therefore \cos c = \cot A \sin a \csc b \cot B \sin b \csc a$

20) $= \cot A \cot B$.

Hence: (8) $\cos c = \cot A \cot B$

21) From (2): $\cos A = \tan b \cot c$

22) $= \dfrac{\sin b}{\cos b} \cdot \dfrac{\cos c}{\sin c}$

23) $= \dfrac{\sin b(\cos a \cos b)}{\cos b \sin c}$ (cf. (1).)

24) $= \dfrac{\sin b \cos a}{\sin c}$.

25) From (5): $\sin c = \dfrac{\sin b}{\sin B}$.

26) Substitute 25) in 24): $\cos A = \dfrac{\sin b \cos a \sin B}{\sin b}$

or $\boxed{\cos A = \sin B \cos a}$

27) Similarly: (10) $\boxed{\cos B = \sin A \cos b}$

**238. Validity of the Ten Formulas.** In order to establish the validity of the Ten Formulas for any spherical triangle $ABC$ in which one angle, $C$, is a right angle it is sufficient to consider two cases: first, when both legs are obtuse; second, when one leg is obtuse and the other is acute.

*Case I.* $C = 90°$, $a > 90°$, $b > 90°$ (Fig. 259).

Extend $\widehat{CA}$ and $\widehat{CB}$ to meet again at $C'$.

Then $\widehat{CAC'} = \widehat{CBC'} = 180°$; $\angle C' = 90°$.
Also, $b' < 90°$, $a' < 90°$.
Moreover, $c < 90°$ (§ 236).

$\therefore \triangle A'C'B'$ is a spherical triangle of the type already discussed in § 237. Therefore each of the Ten Formulas is valid for $\triangle A'B'C'$.

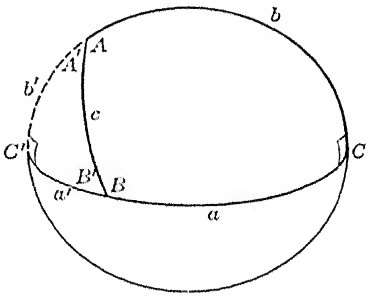



Fig. 259

1) Apply ① to $\triangle A'B'C''$: $\cos c' = \cos a' \cos b'$.
2) But $\cos c' = \cos c$, since $c$ is common to the two triangles.
3) Also, $\cos a' = -\cos a$ and $\cos b' = -\cos b$. (T 5–$e$)
4) $\therefore$ 1) becomes: $\cos c = \cos a \cos b$.
5) $\therefore$ ① is valid for $\triangle ABC$.
6) Apply ② to $\triangle A'B'C''$: $\cos A' = \tan b' \cot c$.
7) But $\cos A' = -\cos A$ and $\tan b' = -\tan b$. (T 5–$e$, 5–$f$)
8) $\therefore$ 6) becomes: $\cos A = \tan b \cot c$.
9) $\therefore$ ② is valid for $\triangle ABC$.
10) The validity of each of the remaining formulas is established in like manner.

*Case II.* $C = 90°$, $a > 90°$, $b < 90°$ (Fig. 260).

Extend $\widehat{BA}$ and $\widehat{BC}$ to meet again at $B'$.

Then $\widehat{BAB'} = \widehat{BCB'} = 180°$; $B' = B$; $C'' = C = 90°$.

Since $a > 90°$, then $a' < 90°$.

By § 236, $c > 90°$. $\therefore c' < 90°$.

$\therefore$ $\triangle A'C'B'$ is a triangle of the type discussed in § 237. Therefore, the Ten Formulas are each valid for $\triangle A'B'C'$.

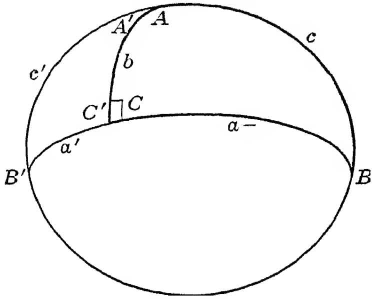



FIG. 260

1) Apply ① to $\triangle A'B'C''$: $\cos c' = \cos a' \cos b$.
2) But $\cos c' = -\cos c$ and $\cos a' = -\cos a$. (T 5–$e$)
3) Substituting in 1): $\cos c = \cos a \cos b$.
4) $\therefore$ ① is valid for $\triangle ABC$.
5) In similar fashion the validity for each of the remaining formulas can be established.

Therefore, the Ten Formulas are valid for any spherical triangle in which one angle is a right angle.

We group the Ten Formulas together for ready reference. For reasons to be made apparent presently, these formulas need *not* be memorized.

| | |
|---|---|
| ① | $\cos c = \cos a \cos b$ |
| ② | $\cos A = \tan b \cot c$ |
| ③ | $\cos B = \tan a \cot c$ |
| ④ | $\sin a = \sin A \sin c$ |
| ⑤ | $\sin b = \sin B \sin c$ |
| ⑥ | $\sin b = \cot A \tan a$ |
| ⑦ | $\sin a = \cot B \tan b$ |
| ⑧ | $\cos c = \cot A \cot B$ |
| ⑨ | $\cos A = \sin B \cos a$ |
| ⑩ | $\cos B = \sin A \cos b$ |

The following theorem is useful in determining whether or not a triangle is possible under certain given conditions.

**239.**

In a spherical triangle $ABC$ with one right angle, $C$: $\sin A$ must be greater than $\sin a$, $\sin B$ must be greater than $\sin b$.

1) From (4): $\sin c = \dfrac{\sin a}{\sin A}$.
2) If $\sin A$ were *equal* to $\sin a$, then $\sin c$ would equal 1, and hence $c$ would equal 90°. In this case, $a$ and $b$ would have to be quadrants, and the given triangle would have to be tri-rectangular. Therefore, if $C$ is the only right angle of the triangle, $\sin A$ cannot equal $\sin a$.
3) If $\sin A$ were *less* than $\sin a$, $\sin c$ would be greater than 1, which is impossible.
4) Therefore, $\sin A$ must be *greater* than $\sin a$, and similarly $\sin B$ must be *greater* than $\sin b$.

The following exercises are extremely important. Each one should be carefully done, and the correct answers should be kept on file. Each exercise, of course, concerns a right spherical triangle having one and only one angle, $C$, a right angle.

## EXERCISES

*Group Thirty-four*

**1.** Discounting angle $C$, which is 90°, there are *five* remaining parts to a triangle. If a formula is to be prepared which involves in turn each possible combination of these parts taken *three* at a time, how many such formulas are necessary?

**2.** Among the Ten Formulas which ones, if any, involve the same three parts?

**3.** Is a part of a spherical triangle uniquely determined if its cosine is known? its tangent? its cotangent? its sine?

**4.** Select the formula which you would use in each of the following instances:

| *Given* | *Find* | *Given* | *Find* |
|---|---|---|---|
| $A, B$ | $c$ | $B, c$ | $a$ |
| $a, b$ | $c$ | $A, B$ | $a$ |
| $b, c$ | $a$ | $a, b$ | $A$ |
| $A, b$ | $B$ | | |

**5.** If $A$ and $c$ are given, is $a$ uniquely determined? (See (4) and § 235.) If $B$ and $c$ are given, is $b$ uniquely determined?

**6.** In each of the following instances state whether or not the part to be found is uniquely determined. Justify your answer.

| *Given* | *Find* | *Given* | *Find* |
|---|---|---|---|
| $b, c$ | $B$ | $B, b$ | $A$ |
| $a, c$ | $A$ | $A, a$ | $b$ |
| $A, a$ | $B$ | $B, b$ | $a$ |

7. The following are given parts, supposedly of a right spherical triangle $ABC$ in which $C = 90°$. State in which instances a triangle cannot exist. (Cf. §§ 203, 235, 236, 239.)

(a) $a = 70°$, $B = 124°$
(b) $B = 60°$, $b = 72°$
(c) $A = 110°$, $b = 120°$
(d) $A = 20°$, $B = 50°$
(e) $c = 50°$, $A = 84°$
(f) $A = 70°$, $a = 95°$
(g) $b = 102°$, $c = 135°$
(h) $A = 108°$, $a = 120°$
(i) $a = 140°$, $b = 130°$, $c = 125°$
(j) $a = 40°$, $b = 108°$, $c = 60°$
(k) $a = 84°$, $b = 100°$

8. If we resolve right triangle solution into cases and classify according to the types of parts given, there are *six* different cases: (1) 2 legs*; (2) 2 angles; (3) leg and adjacent angle; (4) leg and hypotenuse*; (5) hypotenuse and adjacent angle; (6) leg and opposite angle.

Examine each of the above six cases and show that:
(a) in each of the first five cases any part sought will be uniquely determined;
(b) in the last case the part sought may have either of *two* values.

In other words, show that in each of the first five cases there is *one* triangle possible, if any; that in the sixth case there may be *two* triangles possible.

Fig. 261 illustrates the sixth case.

If $\acute{A}\acute{B}$ and $\acute{A}\acute{C}$ are extended to meet again at $A'$, a second rt $\triangle A'CB$ is formed, having $C = 90°$, $A' = A$, and side $a$ the same as for $\triangle ABC$. Either $\triangle A'CB$ or $\triangle ACB$ satisfies the given conditions, viz., that a leg and the angle opposite have certain given values.

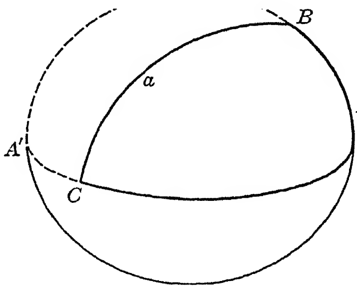



FIG. 261

9. Select the proper formula and solve for the indicated parts, giving results to the nearest minute.

| | *Given* | *Find* |
|---|---|---|
| (a) | $a = 78°$, $b = 62°$ | $c$ |
| (b) | $A = 100°$, $B = 75°$ | $c$ |
| (c) | $B = 67°$, $a = 110°$ | $A$ |
| (d) | $A = 115°$, $b = 80°$ | $a$ |
| (e) | $B = 42°$, $c = 54°$ | $A$ |
| (f) | $A = 70°$, $a = 58°$ | $c$ |
| (g) | $B = 105°$, $b = 120°$ | $a$ |

* In dealing with spherical triangles we use the terms "leg," "hypotenuse," "altitude," "median" as in the case of plane triangles. An "altitude" is a great circle arc drawn from a vertex perpendicular to the opposite side. A "median" is a great circle arc joining a vertex with the mid-point of the opposite side.

**240. *Napier's Rule of Circular Parts.** It is quite unnecessary to memorize the Ten Formulas. Each one can be obtained as needed directly from the diagram of Fig. 262 by using a simple and ingenious device known as Napier's Rule of Circular Parts.

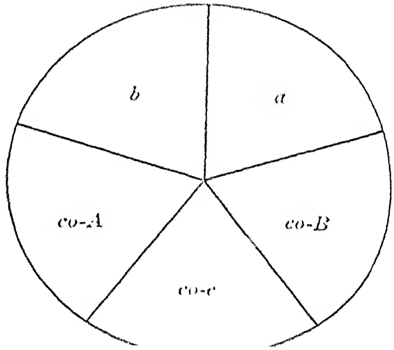



FIG. 262

In the diagram the symbols "co-$A$", "co-$B$", "co-$c$" mean respectively "complement of $A$," "complement of $B$," "complement of $c$." The quantities $b$, $a$, co-$B$, co-$c$, co-$A$ are the "circular parts" mentioned above.

In using the diagram we shall employ the terms "middle part"(*mp*), "adjacent parts"(*adj*), and "opposite parts" (*opp*). If any one of the five quantities is designated as a "middle part," the quantities immediately adjoining this are the two "adjacent parts," and the remaining parts are the two "opposite parts." Thus, if co-$c$ is selected as the *mp* then co-$A$ and co-$B$ are the *adj*, and $b$ and $a$ are the *opp*.

> *Napier's Rule.* The sine of any middle part equals the product of the tangents of the two adjacent parts. The sine of any middle part equals the product of the cosines of the opposite parts.
>
> That is, (1) sin (*mp*) = tan (adj) tan (*adj*).
>
> (2) sin (*mp*) = cos (opp) cos (*opp*).

Each of the Ten Formulas can be written by choosing in turn each of the five quantities of the diagram as a *mp* and then applying part (1) and then part (2) of Napier's Rule.

For example, let co-$c$ = *mp*; then co-$A$ and co-$B$ are the *adj*, $b$ and $a$ are the *opp*.

By part (1): sin (co-$c$) = tan (co-$A$) tan (co-$B$) or cos $c$ = cot $A$ cot $B$, which is actually formula (8).

* John Napier (Lord of Merchiston), Scottish mathematician, was born in 1550 at Merchiston, now a part of Edinburgh. The contribution to mathematics for which he is generally best known was his invention of a system of logarithms. He was the inventor, also, of several ingenious computing machines. His Rule of Circular Parts for the solution of the right spherical triangle appeared in 1614. At one time or another Napier devoted considerable thought to the possible creation of various engines of destruction to be used in warfare, one such device being a sort of war chariot armed with high-powered cannon — an idea which eventually materialized in the form of the modern tank. In addition to his activities in mathematical and scientific research he put forth several works of a theological nature. For the scientific flavor of the ideas advanced in these treatises he was enthusiastically acclaimed a visionary and genius by some, but by others he was damned as a meddler in the Black Art! Napier died in 1617 at the age of 67.

By part (2): $\sin(\text{co-}c) = \cos b \cos a$ or $\cos c = \cos a \cos b$, which is actually formula ①.

*Example 1.* Given: $A = 40°$, $B = 52°$. Find $b$ to the nearest minute.

$A$, $B$, $b$ are the three parts involved. $A$ and $B$ are known; $b$ is to be found.

On the Napier diagram (Fig. 263) select the three sectors containing $A$, $B$, $b$, respectively.

Apply Napier's Rule, using co-$B$ as the *mp*:

$$\sin(\text{co-}B) = \cos(\text{co-}A)\cos b$$

or

$$\cos B = \sin A \cos b. \qquad \text{Hence,} \quad \cos b = \frac{\cos B}{\sin A}$$

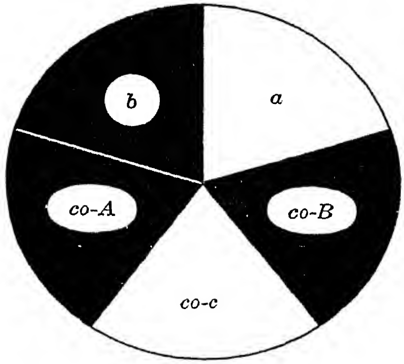



FIG. 263

Substitute the given values for $B$ and $A$, and use logarithms to compute the answer.

$$\cos b = \frac{\cos 52°}{\sin 40°}$$

$$\log \cos 52° = 9.7893 - 10$$
$$\log \sin 40° = 9.8081 - 10$$
$$\log \cos b = 9.9812 - 10$$

$$\boxed{\therefore\ b = 16°\ 45'}$$

*Example 2.* Given: $c = 120°$, $B = 72°$. Find $a$ to the nearest minute.

$c$, $B$, $a$ are the parts involved. Mark these on the diagram (Fig. 264).

Apply Napier's Rule, using co-$B$ as the *mp*:

$$\sin(\text{co-}B) = \tan a \tan(\text{co-}c)$$

or

$$\cos B = \tan a \cot c.$$

$$\tan a = \frac{\cos B}{\cot c}$$

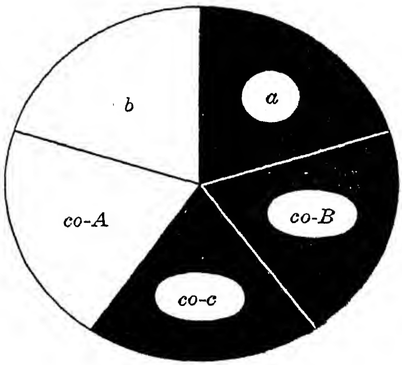



FIG. 264

Note ahead of time that tan $a$ must be *negative* since $B$ is acute and $c$ is obtuse. Hence $a$ itself will have to be *obtuse*.

$$\tan a = \frac{\cos 72°}{\cot 120°}$$

$$\log \cos 72° = 9.4900 - 10$$
$$\log \cot 120° = 9.7614 - 10$$
$$\log \tan a = 9.7286 - 10$$

$$180° - 28°\ 10' = 151°\ 50'.$$

*Example 3.* Given: $A = 112°$, $c = 50°$. Find $a$ to the nearest minute.

Mark $A$, $c$, $a$ on the diagram (Fig. 265).

$$\sin a = \cos(\text{co-}A)\cos(\text{co-}c)$$

or

$$\sin a = \sin A \sin c.$$

Here $a$ must be found from its *sine*. But there can be no ambiguity because $A$ is known to be obtuse and hence $a$, also, must be obtuse (§ 235).

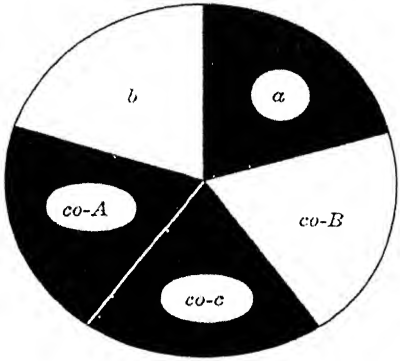



FIG. 265

$$\sin a = \sin 112° \sin 50°$$

$$\boxed{\therefore\ a = 180° - 45°\,16' = 134°\,44'.}$$

$$\begin{aligned} \log \sin 112° &= 9.9672 - 10 \\ \log \sin 50° &= 9.8843 - 10 \\ \hline \log \sin a &= 9.8515 - 10 \end{aligned}$$

## EXERCISES

*Group Thirty-five*

**1.** Derive each of the Ten Formulas by Napier's Rule.

**2.** Set up a Napier diagram for a $\triangle RST$ in which $S = 90°$; for a $\triangle XYM$ in which $X = 90°$; for a $\triangle DAB$ in which $B = 90°$.

**3.** Make a Napier diagram for a $\triangle XYZ$ in which $Z = 90°$. From the diagram obtain a formula which will involve each of the following sets of parts in turn:

(a) $x, y, X$; (b) $X, Y, y$; (c) $z, x, y$; (d) $X, z, x$; (e) $Y, x, y$; (f) $Y, X, z$; (g) $y, X, z$.

**4.** In a $\triangle PQR$, $Q = 90°$, $q = 60°$, $P = 70°$. By Napier's Rule find $r$ to the nearest minute.

**5.** In Ex. 4, find $p$ and $R$ each to the nearest minute.

In Exercises 6–15 assume that $C = 90°$. Find the required part to the nearest minute. Use Napier's Rule throughout.

| Given | Find | Given | Find |
|---|---|---|---|
| **6.** $a = 65°\,10'$, $c = 85°\,30'$. | $A$ | **11.** $a = 56°\,18'$, $c = 39°$. | $b$ |
| **7.** $a = 58°$, $B = 121°$. | $c$ | **12.** $A = 58°$, $c = 110°$. | $b$ |
| **8.** $A = 63°\,12'$, $B = 40°\,15'$. | $b$ | **13.** $A = 60°\,30'$, $B = 136°\,21'$. | $c$ |
| **9.** $a = 80°\,5'$, $b = 40°\,45'$. | $B$ | **14.** $b = 22°\,22'$, $A = 36°\,12'$. | $a$ |
| **10.** $A = 101°\,40'$, $a = 122°$. | $B$ | **15.** $b = 18°\,12'$, $B = 41°\,24'$. | $c$ |

*Example 4.* Given: $A = 75°$, $b = 58°$. Solve the triangle.

Mark the two given parts on the diagram. The first part to be found is then a matter of individual choice.

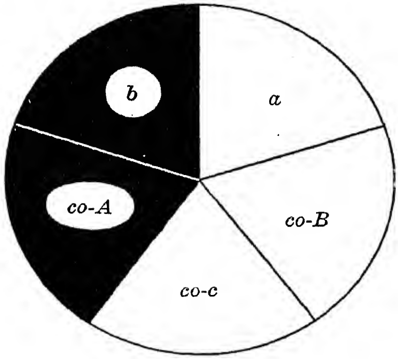



FIG. 266

(1) *Find B:* $\sin(\text{co-}B) = \cos(\text{co-}A)\cos b$

or $\cos B = \sin A \cos b$

or $\cos B = \sin 75° \cos 58°$

$\therefore B = 59° 13'$

$$\begin{aligned} \log \sin 75° &= 9.9849 - 10 \\ \log \cos 58° &= \underline{9.7242 - 10} \\ \log \cos B &= 9.7091 - 10 \end{aligned}$$

(2) *Find a:* In solving for each of the two remaining parts it is usually advisable to obtain a formula which involves the two given parts. Why?

$\sin b = \tan(\text{co-}A) \tan a$

or $\sin b = \cot A \tan a$. Hence: $\tan a = \dfrac{\sin b}{\cot A} = \sin b \tan A$.

$\tan a = \sin 58° \tan 75°$.

Solving by logarithms: $a = 72° 28'$

(3) *Find c:* From the diagram obtain the formula:

$$\cos A = \cot c \tan b.$$

$$\cot c = \frac{\cos A}{\tan b} = \cos A \cot b = \cos 75° \cot 58°$$

$$\therefore c = 80° 49'$$

(4) *Check:* From the diagram obtain a formula involving $B$, $a$, $c$. Substitute the calculated parts in the formula:

$$\cos B = \tan a \cot c$$

Further checks can be obtained, of course, by substituting in formulas involving only one or two of the calculated parts at a time.

*Example 5.* Given: $B = 60°$, $b = 47°$. Solve the triangle.

There are *two* solutions (cf. Ex. 8, Group 34). Care must be taken to keep these two different solutions separate.

(1) *Find a:* Formula: $\sin a = \cot B \tan b$

$\sin a = \cot 60° \tan 47°$

| | |
|---|---|
| $\log \cot 60° = 9.7614 - 10$ | |
| $\log \tan 47° = 0.0303$ | |
| $\log \sin a \;\; = 9.7917 - 10$ | |

Sol. I — $a = 38° 14'$ Sol. II — $141° 46'$

(2) *Find A:* Formula: $\cos B = \sin A \cos b$

$$\sin A = \frac{\cos 60°}{\cos 47°}$$

$\therefore \quad A = 47° 10'$ or $132° 50'$

$\log \cos 60° = 9.6990 - 10$
$\log \cos 47° = 9.8338 - 10$
$\log \sin A \;\; = 9.8652 - 10$

The value $47° 10'$ must belong to Solution I, since $A$ and $a$ must always be of the same species. Hence, we have

Sol. I — $A = 47° 10'$ Sol. II — $A = 132° 50'$

(3) *Find c:* Formula: $\sin b = \sin B \sin c$

$$\sin c = \frac{\sin 47°}{\sin 60°}$$

$\therefore \quad c = 57° 37'$ or $122° 23'$.

$\log \sin 47° = 9.8641 - 10$
$\log \sin 60° = 9.9375 - 10$
$\log \sin c \;\; = 9.9266 - 10$

In Solution I, $a$ and $b$ are of the same species. Therefore, $c$ must be acute (§ 236). Hence the value $57° 37'$ must belong to Sol. I. Also, by § 236, $c$ must be obtuse in Sol. II. Thus, we have:

Sol. I — $57° 37'$ Sol. II — $c = 122° 23'$

The two complete solutions are:

I. $a = 38° 14'$, $A = 47° 10'$, $c = 57° 37'$.
II. $a = 141° 46'$, $A = 132° 50'$, $c = 122° 23'$.

(4) *Check:* Check each solution separately by the method or methods suggested in Example 4.

## EXERCISES

### *Group Thirty-six*

Solve and check each of the following right triangles from the parts given. Assume that $C = 90°$. Obtain results to the nearest minute.

1. $a = 31° 24'$, $b = 50° 30'$
2. $a = 121° 32'$, $c = 64° 17'$
3. $b = 53° 45'$, $A = 35° 20'$
4. $c = 98° 30'$, $B = 106° 10'$
5. $A = 62° 47'$, $B = 134° 26'$
6. $a = 54° 30'$, $B = 34° 50'$
7. $a = 50° 30'$, $A = 62° 24'$
8. $B = 115°$, $c = 80° 40'$
9. $A = 103°$, $a = 117°$
10. $b = 20° 45'$, $c = 150°$
11. $A = 140°$, $B = 99° 10'$
12. $a = 73° 50'$, $c = 82° 25'$
13. $b = 161° 14'$, $c = 135° 16'$
14. $B = 70° 7'$, $c = 117° 30'$

**15.** $a = 24° 50'$, $b = 72° 10'$

**16.** $A = 29° 40'$, $b = 100° 20'$

**17.** $c = 125° 28'$, $a = 54° 50'$

**18.** $b = 40°$, $B = 75°$

**19.** $A = 108° 32'$, $B = 72° 48'$

**20.** $c = 145° 15'$, $a = 155° 30'$

**241. The Isosceles Triangle.** Let $\triangle ABC$ be an isosceles triangle with $a = b$ (Fig. 267). Then $A = B$ (§ 229).

Moreover, if a median $CD$ is drawn, it is readily shown that $\angle ACD = \angle BCD$ and that $\angle ADC = \angle BDC = 90°$.

Thus, the solution of an isosceles triangle can be accomplished by working with either of the symmetric right triangles into which the given triangle can be separated.

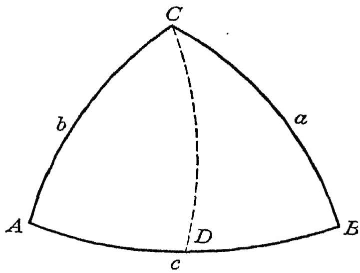



FIG. 267

## EXERCISES

*Group Thirty-seven*

**1.** In $\triangle ABC$, $a = b$. Show exactly how you would solve the triangle, given the parts: (a) $A, C$; (b) $a, B$; (c) $a, A$; (d) $B, c$; (e) $C, c$.

**2.** In which of the cases listed in Ex. 1 is there more than one solution possible?

In Exercises 3–8 solve completely the isosceles $\triangle ABC$ in which $a = b$, giving results to the nearest minute.

**3.** $a = 84°$, $B = 76°$

**4.** $A = 100° 47'$, $C = 88° 19'$

**5.** $B = 65° 40'$, $b = 45°$

**6.** $c = 128°$, $A = 79° 12'$

**7.** $C = 140° 15'$, $B = 82° 10'$

**8.** $c = 96°$, $C = 112°$

**9.** In an isosceles $\triangle ABC$ with $a = b$, if $c$ is obtuse must $C$ be obtuse? Explain carefully.

**242. The Quadrantal Triangle.** A *quadrantal spherical triangle* is a spherical triangle in which one side is a quadrant (90°).

The solution of a quadrantal triangle is accomplished by solving the polar triangle, applying § 230.

For example, let the given quadrantal $\triangle ABC$ have $c = 90°$, $A = 76°$, $B = 52°$. Then in the polar $\triangle A'B'C'$, $C' = 90°$, $a' = 104°$, $b' = 128°$. $\triangle A'B'C'$, being of the type discussed in this chapter, can now be solved in the usual way. Apply § 230 to the solution of $\triangle A'B'C'$ to obtain the solution of the given $\triangle ABC$.

## EXERCISES

*Group Thirty-eight*

Solve each of the following quadrantal triangles, given:

**1.** $c = 90°$, $A = 122°$, $b = 59°$

**2.** $c = 90°$, $A = 85°$, $B = 110°$

**3.** $c = 90°$, $C = 131° 10'$, $a = 118° 10'$

**4.** $c = 90°$, $C = 67° 29'$, $B = 58° 42'$

**5.** $c = 90°$, $a = 120° 50'$, $A = 114° 37'$

**6.** $c = 90°$, $A = 98° 6'$, $B = 132° 40'$

# *Chapter Sixteen*

## SPHERICAL TRIGONOMETRY: THE SPHERICAL OBLIQUE TRIANGLE

In Chapter Fifteen we evolved means of solving any spherical *right* triangle, *isosceles* triangle, or *quadrantal* triangle. The present chapter introduces the study of the spherical *oblique* triangle, that is, a triangle in which no angle is 90°. In a sense, the preceding chapter is sufficient for most practical needs, since any oblique triangle can always be resolved into two right triangles. However, the solution of an oblique triangle by such means is awkward and cumbersome. Therefore, we shall derive two laws which are specifically adapted to solving oblique triangles: the *Law of Cosines* and the *Law of Sines*.

In the work of this chapter the following relations of parts are helpful as a means of checking the validity of results obtained.

**243.** In any spherical triangle $ABC$:

I. If $a > b$, then $A > B$, and conversely (§ 229).
II. $a + b + c < 360°$ (§ 203–B).
III. $a + b > c$ (§ 203–C).
IV. $540° > A + B + C > 180°$ (§ 203–D).
V. $180° + C > A + B$ (Ex. 7, Group 21).
VI. One-half the sum of any two sides must be of the same species as one-half the sum of the two angles opposite these sides. (To be proved in § 255.)

**244. The Law of Cosines.** Let $\triangle ABC$ be any oblique spherical triangle. Draw the altitude $CD$ ($h$).

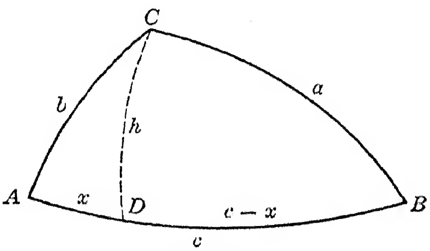


FIG. 268

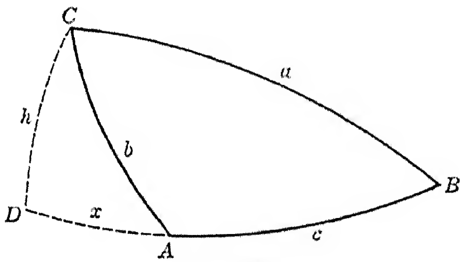


FIG. 269

PART I. $h$ falls within $\triangle CAB$ (Fig. 268.)

1) In $\triangle CDB$: $\cos a = \cos (c - x) \cos h.$ ( ① )

2) $= (\cos c \cos x + \sin c \sin x) \cos h$ (T 6–$b$)

3) $= \cos c (\cos x \cos h) + \sin c (\sin x \cos h)$

4) $= \cos c (\cos x \cos h) + \sin c \tan x (\cos x \cos h).$ (T 3–$a$)

5) But in $\triangle ADC$: $\cos x \cos h = \cos b$ ( ① )
and $\tan x = \cos A \tan b.$ ( ② )

6) Substituting in 4):

$$\cos a = \cos c \cos b + \sin c (\cos A \tan b) \cos b.$$

7) or ⑪ $\cos a = \cos b \cos c + \sin b \sin c \cos A$ (T 3–$a$)

PART II. $h$ falls outside $\triangle CAB$ (Fig. 269).

8) Proceed as in Part I, noting that $\cos CAD = - \cos CAB.$ (T 5–$e$)
Corresponding formulas for $\cos b$ and $\cos c$ are similarly derived.

PART III (Either figure).

9) Let $\triangle A'B'C'$ be polar to $\triangle ABC$.
Then $a' = 180° - A,\ b' = 180° - B,\ c' = 180° - C,\ A' = 180° - a.$

10) Apply ⑪ to the polar triangle $A'B'C'$:

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

11) or $- \cos A = (- \cos B)(- \cos C) + \sin B \sin C (- \cos a)$

12) or ⑭ $\cos A = - \cos B \cos C + \sin B \sin C \cos a$

Corresponding formulas for $\cos B$ and $\cos C$ are similarly derived.

Thus, we have the following six formulas which comprise the Law of Cosines of Spherical Trigonometry. The first three are often known as the *Law of Cosines for Sides;* the last three are the *Law of Cosines for Angles.*

| | |
|---|---|
| ⑪ | $\cos a = \cos b \cos c + \sin b \sin c \cos A$ |
| ⑫ | $\cos b = \cos a \cos c + \sin a \sin c \cos B$ |
| ⑬ | $\cos c = \cos a \cos b + \sin a \sin b \cos C$ |
| ⑭ | $\cos A = - \cos B \cos C + \sin B \sin C \cos a$ |
| ⑮ | $\cos B = - \cos A \cos C + \sin A \sin C \cos b$ |
| ⑯ | $\cos C = - \cos A \cos B + \sin A \sin B \cos c$ |

*Use of the Formulas*

*Example 1.* In $\triangle ABC$: $a = 40°, b = 60°, C = 70°.$ Find $c$ to the nearest minute.

(13) $\cos c = \cos a \cos b \qquad \underbrace{\sin a \sin b \cos C}_{y}$

$$\begin{aligned} \log \cos 40° &= 9.8843 - 10 \\ \log \cos 60° &= 9.6990 - 10 \\ \log x &= 9.5833 - 10 \end{aligned} \qquad \begin{aligned} \log \sin 40° &= 9.8081 - 10 \\ \log \sin 60° &= 9.9375 - 10 \\ \log \cos 70° &= 9.5341 - 10 \\ \log y &= 9.2797 - 10 \end{aligned}$$

$$x = 0.3831 \qquad \therefore \quad y = 0.1904$$

$$\therefore \quad \cos c = x + y = 0.5735$$

$$= 55° 1'$$

*Example 2.* In $\triangle ABC$: $a = 50°$, $b = 62°$, $c = 71°$. Find $A$ to the nearest minute.

(11) $\cos a = \cos b \cos c + \sin b \sin c \cos A$

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\cos a - m}{D}$$

$(m)$

$$\begin{aligned} \log \cos 62° &= 9.6716 - 10 \\ \log \cos 71° &= 9.5126 - 10 \\ \log m &= 9.1842 - 10 \end{aligned}$$

$$\therefore \quad m = 0.1528$$

$(D)$

$$\begin{aligned} \log \sin 62° &= 9.9459 - 10 \\ \log \sin 71° &= 9.9757 - 10 \\ \log D &= 9.9216 - 10 \end{aligned}$$

$(\cos A)$

$$\begin{aligned} \cos 50° &= 0.6428 \\ m &= 0.1528 \\ \text{Numerator} &= 0.4900 \\ \log N &= 9.6902 - 10 \\ \log D &= 9.9216 - 10 \\ \therefore \log \cos A &= 9.7686 - 10 \end{aligned}$$

$$A = 54° 4'$$

---

*Note:* In case one or more of the parts given is *obtuse*, remember that the cosines of such quantities are *negative*, and hence take care to allow for this in doing the computation.

For example, suppose that in solving a $\triangle ABC$ for $a$ you use the formula: $\cos a = \cos b \cos c + \sin b \sin c \cos A$. Suppose that $b < 90°$ and that $c > 90°$ and $A > 90°$. Then $\cos b$, $\sin b$, $\sin c$ are each positive, but $\cos c$ and $\cos A$ are each negative. Hence, $\cos a$ will be negative, and $a$ itself will be obtuse.

$$\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A \\ &= (+)(-) \quad + \quad (+)(+)(-) \\ &= (-) + (-) \\ &= (-) \end{aligned}$$

## EXERCISES

### *Group Thirty-nine*

In the numerical problems solve for the required parts to the nearest minute.

1. Given: $b = 62° 48'$, $c = 81° 20'$, $A = 50°$. Find $a$.
2. Given: $a = 120° 54'$, $c = 88°$, $B = 46° 36'$. Find $b$.
3. Given: $A = 130° 22'$, $C = 105° 30'$, $b = 125° 15'$. Find $B$.
4. Given: $A = 25° 42'$, $B = 40° 42'$, $c = 152° 32'$. Find $C$.

5. Given: $a = 23°$, $b = 62°$, $c = 71°$. Find $A$ and $B$.

6. Given: $A = 65°$, $B = 78°$, $C = 58°$. Find $b$ and $c$.

7. Given: $a = 100°$, $b = 110°$, $c = 115°$. Find $B$.

8. Given: $A = 121°$, $B = 117° 30'$, $C = 135° 15'$. Find $c$.

9. Given: $a = 67° 32'$, $b = 110° 40'$, $C = 52° 18'$. Find $c$ and $A$.

10. Given: $A = 125° 12'$, $B = 105° 30'$, $c = 120° 34'$. Find $C$ and $b$.

11. In $\triangle ABC$, $\overset{\frown}{CD}$ is a median. $a = 50°$, $B = 62°$, $c = 80°$. Find:
(a) the number of degrees in $\overset{\frown}{CD}$;
(b) the length of $\overset{\frown}{CD}$ if the radius of the sphere is 10 inches;
(c) the number of degrees in $\angle CDB$.

12. In $\triangle ABC$, $\overset{\frown}{CD}$ bisects $C$ and meets side $c$ at $D$. $C = 100°$, $\overset{\frown}{CD} = 78°$, $b = 57°$. Find $\overset{\frown}{AD}$ and $A$.

13. In $\triangle ABC$, $\overset{\frown}{CD}$ bisects $C$ and meets side $c$ at $D$. $b = 70°$, $\overset{\frown}{AD} = 53°$, $\overset{\frown}{CD} = 62°$. Find $C$.

14. In Fig. 270, $\overset{\frown}{CD}$ bisects $C$. Let $\angle CDA = \phi$.

Prove: $$\cos \tfrac{1}{2} C = \frac{\cos B - \cos A}{2 \cos \phi}.$$

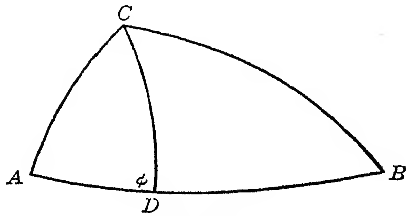



FIG. 270

15. Prove that in any spherical $\triangle ABC$:

$$\cos c = \frac{\cos a \cos b - \sin a \sin b \cos A \cos B}{1 - \sin a \sin b \sin A \sin B}$$

(Solve for $\cos C$ in formulas (13) and (16). Equate the two expressions for $\cos C$, and solve for $\cos c$.)

**245. The Law of Sines.** Let $\triangle ABC$ be any oblique spherical triangle.
Draw the altitude $h$.

PART I. $h$ lies within $\triangle ABC$ (Fig. 271).

1) In $\triangle CDA$: $\sin A \sin b = \sin h$
In $\triangle CDB$: $\sin B \sin a = \sin h$ ( (4) )

2) $\therefore \sin A \sin b = \sin B \sin a$.

3) or $\dfrac{\sin b}{\sin B} = \dfrac{\sin a}{\sin A}$.

4) By drawing another altitude prove: $\dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C}$.

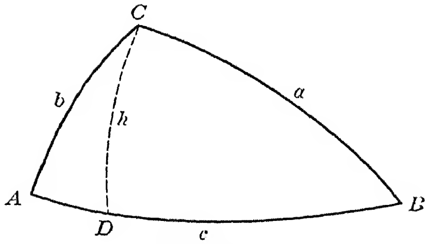


FIG. 271

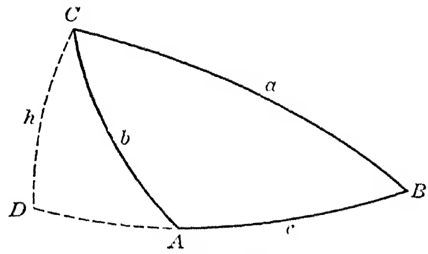


FIG. 272

PART II. $h$ lies outside $\triangle ABC$ (Fig. 272).

5) Proceed as in Part I, noting that $\sin \angle CAD = \sin \angle CAB$. (T 5–$d$)

Therefore, we have: $$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

The Law of Sines does not possess the precision of the Law of Cosines in that the part to be found by this formula is characterized by its *sine*, and may, therefore, have *two* possible values. In practical application, especially in problems having to do with the Earth's surface, it is generally known beforehand whether the part sought is acute or obtuse; and hence the use of the Law of Sines leads to no ambiguity. If, however, some test *is* needed to determine whether the part sought is acute or obtuse, one or more of the relations of § 243 may be applied.

Examples 3 and 4 illustrate the use of the Law of Sines in conjunction with the Law of Cosines.

*Example 3.* Given: $a = 142°$, $b = 68°$, $C = 147°$. Solve the triangle.

1) *Find c.*

By Law of Cosines:
$$\begin{aligned} \cos c &= \cos a \cos b + \sin a \sin b \cos C \\ &= (-)(+) \;+\; (+)(+)(-) \\ &= (-) \;+\; (-) \\ &= (-) \end{aligned}$$

Solving: $c = 140° \, 43'$.

2) *Find A.* Use Law of Sines:

$$\sin A = \frac{\sin a \sin C}{\sin c} = \frac{\sin 142° \sin 147°}{\sin 140° \, 43'}$$

$A = 31° \, 58'$ or $148° \, 2'$

| | |
|---|---|
| log sin 142° = | 9.7893 − 10 |
| log sin 147° = | 9.7361 − 10 |
| | 9.5254 − 10 |
| log sin 140° 43′ = | 9.8015 − 10 |
| log sin $A$ = | 9.7239 − 10 |

By § 243–I: $a > c$; $\therefore A > C$. $\therefore$ discard the value $31° \, 58'$

$$A = 148° \, 2'.$$

3) *Find B.* Use Law of Sines:

$$\sin B = \frac{\sin b \sin C}{\sin c} = \frac{\sin 68° \sin 147°}{\sin 140° 43'}$$

$$\begin{aligned} \log \sin 68° &= 9.9672 - 10 \\ \log \sin 147° &= 9.7361 - 10 \\ &\quad 9.7033 - 10 \\ \log \sin 140° 43' &= 9.8015 - 10 \\ \log \sin B &= 9.9018 - 10 \end{aligned}$$

$B = 52° 54'$ or $127° 6'$

By § 243–V: $180° + B > A + C$. $\therefore$ discard the value $52° 54'$

$$B = 127° 6'.$$

4) *Check.* Use any of the formulas (11)–(17) which involve the parts obtained.

*Example 4.* Given: $A = 150°$, $B = 123°$, $c = 137°$. Solve the triangle.

1) *Find C.*
By the Law of Cosines: 
$$\begin{aligned} \cos C &= -\cos A \cos B + \sin A \sin B \cos c \\ &= -\quad (-)(-) \quad + \quad (+)(+)(-) \\ &= \quad (-) \end{aligned}$$

Solving: $C = 141° 7'$.

2) *Find a.* Use the Law of Sines:

$$\sin a = \frac{\sin A \sin c}{\sin C} = \frac{\sin 150° \sin 137°}{\sin 141° 7'}$$

Solving by logarithms: $a = 32° 55'$ or $147° 5'$.
By § 243–I: $A > C$; $\therefore a > c$. $\therefore$ discard the value $32° 55'$.

$$a = 147° 5'.$$

3) *Find b.* Use the Law of Sines:

$$\sin b = \frac{\sin B \sin c}{\sin C} = \frac{\sin 123° \sin 137°}{\sin 141° 7'}$$

Solving by logarithms: $b = 65° 42'$ or $114° 18'$.
By § 243–II: $a + b + c < 360°$. $\therefore$ discard the value $114° 18'$.

$$b = 65° 42'.$$

4) *Check.* See Example 3.

*Note:* In problems like the preceding, when the last two parts are being found it may be expedient to postpone the testing of values by means of § 243 until both parts are calculated. Thus, in Example 4, if we had solved for $b$ before solving for $a$, there would have been no direct way of making the correct choice of values.

In Examples 3 and 4 we could have used the Law of Cosines for finding each part if we had wished. But quite obviously the Law of Sines is simpler to manipulate when § 243 is at our disposal for testing results.

**246. The Six Cases of Oblique Triangle Solution.** Chapter Thirteen gave us the following facts:

On any given sphere two spherical triangles with corresponding parts similarly ordered are congruent if:

(1) two sides and the included angle of one respectively equal
two sides and the included angle of the other (Ex. 9, Group 21);

(2) two angles and the included side of one respectively equal two angles and the included side of the other (Ex. 10, Group 21);

(3) three sides of one equal three sides of the other, respectively (§ 206–*B*);

(4) three angles of one equal three angles of the other, respectively (Ex. 12, Group 21).

In consequence, we may state:

*One* and *only one* spherical triangle (if any at all) is determined if any one of the following combinations of parts is given:

| | | |
|---|---|---|
| *Case 1.* | Two sides and the included angle. | (SAS) |
| *Case 2.* | Two angles and the included side. | (ASA) |
| *Case 3.* | Three sides. | (SSS) |
| *Case 4.* | Three angles. | (AAA) |

A study of Examples 1–4 just presented reveals that the Law of Cosines and the Law of Sines together with § 243 are sufficient for our needs in solving an oblique spherical triangle falling under any one of these four cases.

The two remaining cases are:

| | | |
|---|---|---|
| *Case 5.* | Two sides and the angle opposite one of these. | (SSA) |
| *Case 6.* | Two angles and the side opposite one of these. | (AAS) |

In each of the cases, 5 and 6, there may be *one* or there may be *two* triangles (if any at all) which will satisfy the given conditions. It is quite possible to deal with these cases immediately if it is desired. For example, in Case 5, suppose that we are given $a$, $b$, $B$ of a $\triangle ABC$. Using §§ 245 and 243 we can determine the value or values of $A$. Next, we can use the formula of Ex. 15, Group Thirty-nine to determine the corresponding value or values of $c$. Finally, we can use § 244 to determine $C$. The last two parts, $c$ and $C$, can also be found in the following way: drop an altitude from vertex $C$, thus creating two right triangles; work with these right triangles, and combine results to obtain $c$ and $C$. But it is perhaps advisable to postpone a discussion of the complete triangle solution in Cases 5 and 6 until the supplementary formulas of Chapter Eighteen are acquired.

## EXERCISES

### *Group Forty*

In Exs. 1–16 solve the triangle completely from the parts given. Obtain results to the nearest minute.

1. $a = 73°$, $b = 46°$, $C = 32°$.

2. $a = 40°$, $b = 50°$, $c = 43°$.

3. $A = 59° 17'$, $B = 76° 11'$, $C = 80° 32'$.

4. $A = 76° 22'$, $B = 42°$, $c = 106° 12'$.

5. $A = 87°$, $B = 74°$, $C = 96°$.

6. $a = 64° 16'$, $b = 57° 12'$, $c = 100°$.

7. $a = 126°$, $b = 151°$, $C = 140°$.

8. $B = 30° 30'$, $C = 121° 25'$, $a = 41°$.

9. $b = 100°$, $c = 58°$, $A = 108°$.

10. $A = 34° 35'$, $B = 39° 6'$, $C = 136° 45'$.

11. $a = 105° 10'$, $b = 50° 20'$, $c = 62° 28'$.

12. $A = 132° 4'$, $B = 53° 17'$, $c = 72° 5'$.

13. $a = 118° 17'$, $c = 71° 14'$, $B = 51° 40'$.

14. $A = 118° 5'$, $B = 128° 10'$, $C = 78° 40'$.

15. $a = 62° 8'$, $b = 41° 30'$, $c = 98° 42'$.

16. $C = 110° 15'$, $A = 146° 10'$, $b = 125°$.

In Exs. 17–23 use the Law of Sines to find the part required. Refer to § 243 to determine the number of solutions in each instance.

| | *Given* | *Find* |
|---|---|---|
| 17. | $b = 100°$, $c = 65°$, $B = 97°$ | $C$ |
| 18. | $A = 133° 9'$, $C = 141° 36'$, $c = 126° 41'$ | $a$ |
| 19. | $a = 42°$, $b = 119°$, $A = 30°$ | $B$ |
| 20. | $B = 21° 36'$, $C = 145° 12'$, $c = 135° 45'$ | $b$ |
| 21. | $a = 125° 30'$, $b = 68° 42'$, $A = 57° 10'$ | $B$ |
| 22. | $A = 72° 50'$, $B = 82°$, $a = 123°$ | $b$ |
| 23. | $C = 62°$, $B = 142°$, $b = 152°$ | $c$ |

# *Chapter Seventeen*

## SPHERICAL TRIGONOMETRY: APPLICAT

In the present chapter* we shall apply Spherical Trigonometry to the solutions of such problems as finding the spherical distance between two known points on the Earth's surface, determining the positions and courses of ships which are assumed to be sailing along the arcs of great circles, and so on. At the end of the chapter a brief discussion of the celestial sphere and its relation to the Earth reveals further applications of spherical triangle solution. Before commencing the study of Chapter Seventeen review § 197 in which such terms as *latitude*, *longitude*, *nautical mile*, *knot* are defined.

**247. Ship's Course. True Bearing.** Let $NAS$ be the meridian through a point $A$; let the ship's path from $A$ be the great circle arc $AB$ (Figs. 273–275). The *ship's course from* $A$ is denoted by the spherical angle $NAB$. Thus, in Fig. 273 the ship's course from $A$ is 60°; in Fig. 274 the course is 130°; in

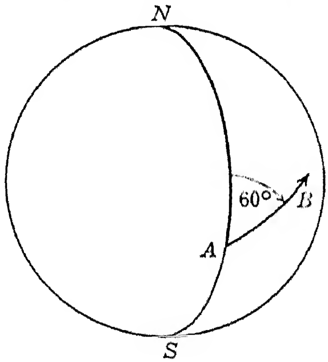



FIG. 273

Fig. 275 the course is 250°. The angle defining the ship's course from $A$ has for its initial side the meridian arc $AN$; the angle is generated in a clockwise sense, having for its terminal side the arc $AB$.

* If it is thought desirable for the student to acquire more extensive technique in triangle solution, the formulas of §§ 251–255 of Chapter Eighteen may be studied before taking up the applications of Chapter Seventeen. These supplementary formulas shorten the computation in some instances; and in all cases they are ideally adapted to logarithmic computation.

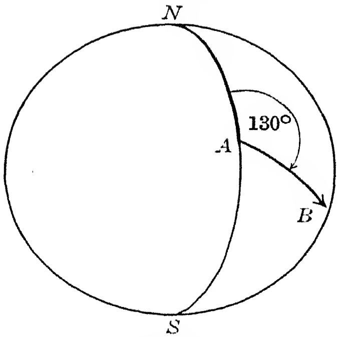



FIG. 274

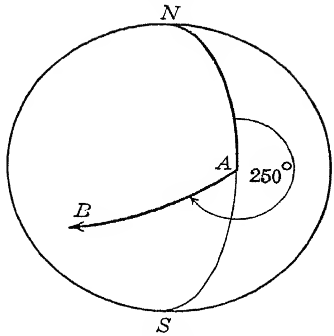



FIG. 275

If $B$ is a point on the ship's path, the angle $NAB$ is also called the *true bearing* of $B$ from $A$.

In the preceding illustrations the ship's course from $A$, or the true bearing of $B$ from $A$, can be expressed as follows: in Fig. 273: N 60° E; in Fig. 274: S 50° E; in Fig. 275: S 70° W. But this method of defining course or bearing is not so common in modern practice as the method first presented.

*Example 1.* $A$ and $B$, two points on the Earth's surface, have the following positions: $A$ (lat. 30° N, long. 80° W), $B$ (lat. 40° N, long. 50° W).
(a) Find the distance $AB$ in nautical miles.
(b) If a ship sails the great circle route from $A$ to $B$ find the ship's course from $A$, i.e., the ship's starting course.

In Fig. 276, $N$ and $S$ are respectively the north and south poles; $g$ is the Meridian of Greenwich (prime meridian); $\widehat{NAS}$ and $\widehat{NBS}$ are the meridians through $A$ and $B$, respectively; $e$ is the equator.

$\widehat{EA} = 30°$; $\therefore$ $\widehat{AN} = 60°$.
$\widehat{FB} = 40°$; $\therefore$ $\widehat{BN} = 50°$.
$\widehat{ET} = 80°$ and $\widehat{FT} = 50°$. $\therefore$ $\widehat{EF} = 30°$.
That is, sph $\angle ANB = 30°$.

*Note.* Arcs $AN$ and $BN$ are called the *co-latitudes* of $A$ and $B$, respectively.
Sph $\angle ANB$ is called the *difference in longitude* for $A$ and $B$.

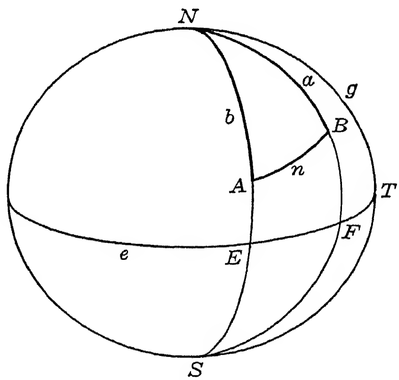



FIG. 276

Our problem seems to be concerned with sph $\triangle ANB$ in which $b = 60°$, $a = 50°$, $N = 30°$. We are to find the length of $n$ in nautical miles and the number of degrees in angle $NAB$.

By § 244, $\quad \cos n = \cos a \cos b + \sin a \sin b \cos N.$
Solving: $\quad n = 26° 22'.$

But one nautical mile is the length of one minute of arc of a great circle (§ 197).

$$\therefore \quad \widehat{AB} = 26(60) + 22 = 1582 \text{ nautical miles.}$$

By § 245, $\quad \sin A = \dfrac{\sin a \sin N}{\sin n}$. Here $A$ must be acute.

Solving: $\quad A = 59° 38'$.

$\therefore$ the ship's starting course is 59° 38′.

In practice it is usually apparent whether $A$ is acute or obtuse. Hence, for the most part no ambiguity results from finding $A$ by the Law of Sines. When $A$ turns out to be nearly 90°, however, it may be uncertain whether $A$ is actually acute or obtuse. In this case the following test may be made. Set up a right triangle having as legs the given value of $\widehat{NA}$ and the computed value of $\widehat{AB}$. Solve for the hypotenuse $h$ by Napier's Rule ($\cos h = \cos \widehat{NA} \cos \widehat{AB}$). If $h$ turns out to be *greater* than the given value of $\widehat{NB}$, then $A$ must be *acute*; if $h$ is *less* than the given value of $\widehat{NB}$, then $A$ must be *obtuse*.

If in Example 1 the arc $AB$ is interpreted as a ship's route, then, as has already been stated, the angle $NAB$ is the ship's course out of $A$ or the ship's *starting course* if the ship is moving from $A$ to $B$. The ship's course at $B$, i.e., the ship's *finishing course* is the angle formed by $\widehat{NB}$ and the prolongation of $\widehat{AB}$. In Example 1 the ship's finishing course, therefore, is the supplement of angle $NBA$. If the ship sails from $B$ to $A$ its starting course is the reflex angle $NBA$, and its finishing course is the reflex angle formed by $\widehat{NA}$ and the prolongation of $\widehat{BA}$.

In the example just discussed points $A$ and $B$ were chosen in the northern hemisphere. Had $A$ and $B$ both been in the southern hemisphere (south of the equator) the usual procedure would be to work with $\triangle ASB$ instead of $\triangle ANB$.

In each of the following problems assume that all distances and ship routes are great circle arcs. Disregard any change of time as the ships travel from one locality to another. Obtain distances in nautical miles, giving results to four significant figures. Find angles or arcs to the nearest minute.

## EXERCISES

### *Group Forty-one*

**1.** Given: $A$ (lat. 10° 30′ N, long. 20° 15′ W), $B$ (lat. 70° 25′ N, long. 65° W). Find: (a) distance $AB$; (b) ship's course from $A$ if it sails from $A$ to $B$.

**2.** Do the same as in Ex. 1, assuming $A$ and $B$ as follows: $A$ (lat. 10° 20′ N, long. 130° E), $B$ (lat. 41° 23′ N, long. 124° W).

**3.** Find the distance from San Francisco (lat. 37° 32′ N, long. 122° 13′ W) to a point near Pearl Harbor (lat. 21° 20′ N, long. 158° W).

**4.** Find the distance from Los Angeles (lat. 33° 43′ N, long. 118° 15′ W) to Manila (lat. 14° 34′ N, long. 120° 57′ E).

**5.** Find the distance from Dakar (lat. 14° 40′ N, long. 17° 25′ W) to Halifax, N.S. (lat. 44° 35′ N, long. 63° 28′ W). Find the true bearing of Halifax from Dakar.

6. Find the distance from Santa Barbara, Cal. (lat. 34° 24′ N, long. 119° 43′ W) to Sydney Bay, Australia (lat. 29° 4′ S, long. 167° 58′ E).

7. Find the distance $AB$ if $A$ (lat. 32° N, long. 120° W) and $B$ (lat. 45° S, long. 75° W) are the given points.

8. Given: $A$ (lat. 60° S, long. 65° W), $B$ (lat. 30° S, long. 60° E). Find $AB$.

9. A cruiser at Portsmouth, N.H. (lat. 43° 4′ N, long. 70° 44′ W) has a rendezvous with other warships at a point $A$ (lat. 30° 20′ N, long. 30° 30′ W) at 10 : 00 P.M. on May 6. If the cruiser can average 30 knots, at what time should it leave Portsmouth in order to reach its destination exactly on time? What course should the cruiser set out of Portsmouth?

10. A ship's course from $A$ (lat. 45° N, long. 60° 10′ W) is 120°. The ship averages 25 knots. Find the ship's position 60 hours later when it has reached a point $B$.

11. In Ex. 10 find the ship's course at $B$.

12. A ship averaging 20 knots leaves Boston (lat. 42° 23′ N, long. 71° 4′ W) at 6 : 00 P.M. on June 1. Its starting course is 110°. Find the ship's position and course 48 hours later.

13. A ship sails from $A$ (lat. 0°, long. 50° W) to a point $B$ (lat. 50° 10′ N, long. 10° W). Find the distance $AB$. (Extend $\widehat{NB}$ to cut the equator at a point $C$. Note that $\triangle ACB$ is a right triangle.)

14. In Ex. 13, if the ship returns from $B$ to $A$ what is its course from $B$?

15. In Ex. 14 find the ship's latitude when its longitude is 20° W.

16. In Ex. 14 find the ship's longitude when it reaches latitude 15° N.

17. In Ex. 14 find the position of the point which is mid-way on the route from $B$ to $A$.

18. Given: $A$ (lat. 36° 50′ N, long. 76° 12′ W), $B$ (lat. 48° 20′ N, long. 5° 10′ W). Find: (a) distance $AB$; (b) starting course of a ship sailing from $A$ to $B$; (c) finishing course of the same ship.

19. In Ex. 18 draw $\widehat{NV}$ perpendicular to $\widehat{AB}$ meeting $\widehat{AB}$ at $V$. Calculate the position of $V$ (Fig. 277).

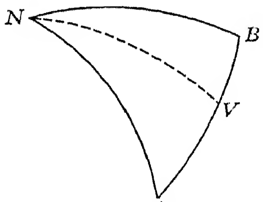



FIG. 277

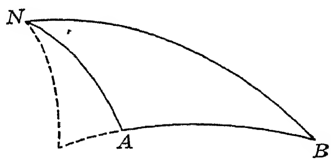



FIG. 278

*Note.* The point $V$ of Ex. 19 is called the *vertex* of the great circle route from $A$ to $B$. The altitude $\widehat{AV}$ of course separates $\triangle ANB$ into two right triangles, and is thus an aid in calculating the position of any chosen point on $\widehat{AB}$. It often happens that $V$ does not lie between $A$ and $B$ but on $\widehat{AB}$ extended (either through $A$ or through $B$). But this situation offers no difficulty to the student who is familiar with spherical triangle solution. See Fig. 278.

**20.** In Ex. 18 find the latitude of the ship when its longitude is 50° W. (Make use o rt $\triangle NVA$ or rt $\triangle NVB$ of Fig. 277.)

**21.** In Ex. 1 find the position (lat. and long.) of $V$.

**22.** In Ex. 4 find the position of $V$.

**23.** In Ex. 8 find the positon of $V$.

**24.** Two ships, $A$ and $B$, leave a point $P$ (lat. 0°, long. 50° W) at the same moment, $B$ sailing two-thirds as fast as $A$. $A$ leaves $P$ on a course of 30°; $B$'s course from $P$ is 338°. After a certain length of time the longitude of $A$ is found to be 20° W. Find the latitude of $A$.

**25.** In Ex. 24 find the distance between the two ships at the time required.

**26.** A ship's course from $A$ (lat. 5° S, long. 135° W) is 328°. Find the longitude of the point where the ship crosses the equator.

**27.** In Ex. 26 find the ship's course at the point where it crosses the equator.

**28.** In Ex. 26, when the ship reaches a point $B$ on its route the ship's course is found to be 290°. What is the latitude of $B$?

**29.** A destroyer sailing at an average rate of 30 knots leaves New York (lat. 40° 28 N, long. 74° W) on a course of 102°. It continues on this route for 24 hours until it reaches a point $A$. At $A$ the ship takes a course of 20° and continues on this second route for 30 hours. At the end of the 30 hours how far is the destroyer from New York?

**30.** Find the position of a point in the Atlantic Ocean which is 1920 nautical miles from Nantucket (lat. 41° 23′ N, long. 70° W) and the same distance from Vera Cruz (lat. 19° 11′ N, long. 96° 4′ W).

**248. The Cosine-Haversine Formula.** In finding the great circle distance between two points on the Earth's surface when the positions of those points are known we used the Law of Cosines. (See § 247.) The computation of this distance can be materially lessened by using a formula which involves *haversines*, and which is deduced from the Law of Cosines formula. The derivation and use of this new formula will now be discussed.

The haversine of an angle $\theta$, hav $\theta$, is $\dfrac{1-\cos\theta}{2}$.

*Derivation of Formula* (Fig. 279.) Let $A$ (lat. $d_1$° N, long. $L_1$° W) and $B$ (lat. $d_2$° N, long. $L_2$° W) be two points on the Earth's surface.

Then $\widehat{EA} = d_1$ and $\widehat{FB} = d_2$.
$N = \widehat{EF} = L_1 - L_2$.
We are seeking a formula for $n$.

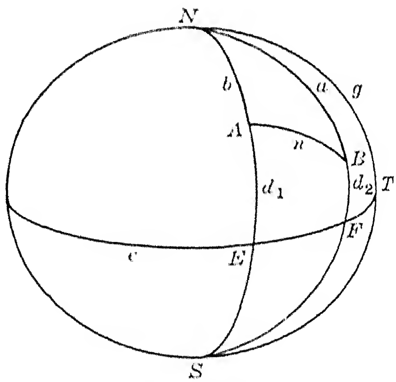


FIG. 279

By the Law of Cosines:

1) $$\cos n = \cos b \cos a + \sin b \sin a \cos N$$

2) $$= \pm \sin d_1 \sin d_2 + \cos d_1 \cos d_2 \cos N, \qquad \text{(T 5–}a\text{, 5–}b\text{)}$$

the sign before the term ($\sin d_1 \sin d_2$) being + or − accordingly as points $A$ and $B$ are both on the *same* side of the equator or on *opposite* sides of the equator.

3) $$\therefore \text{hav } n = \frac{1 - \cos n}{2} = \frac{1 \mp \sin d_1 \sin d_2 - \cos d_1 \cos d_2 \cos N}{2}$$

4) $$\text{or hav } n = \frac{1 - \cos d_1 \cos d_2 \mp \sin d_1 \sin d_2 + \cos d_1 \cos d_2 - \cos d_1 \cos d_2 \cos N}{2}$$

5) $$= \frac{1 - \cos (d_1 \mp d_2) + \cos d_1 \cos d_2 (1 - \cos N)}{2}$$

6) $$= \frac{1 - \cos (d_1 \mp d_2)}{2} + \cos d_1 \cos d_2 \frac{(1 - \cos N)}{2}$$

7) $$\text{or hav } n = \text{hav } (d_1 \mp d_2) + \cos d_1 \cos d_2 \text{ hav } N$$
$$\text{or hav } n = \text{hav } (d_2 \mp d_1) + \cos d_1 \cos d_2 \text{ hav } N \qquad \text{(T 5–}h\text{)}$$

Here we have treated $d_1$ and $d_2$ as purely numerical quantities. If, however, we agree to call the latitude of a point *positive* if the point is *north* of the equator, and *negative* when the point is *south* of the equator, then step 7 may be written:

8) $$\text{hav } n = \text{hav } (d_1 - d_2) + \cos d_1 \cos d_2 \text{ hav } N,$$

where $(d_1 - d_2)$ now represents the *algebraic difference* of $d_1$ and $d_2$. In navigation, the symbol $\sim$ is often used for this purpose instead of the symbol −. The formula may now be written:

9) $$\text{hav } n = \text{hav } (d_1 \sim d_2) + \cos d_1 \cos d_2 \text{ hav } N$$

This is known as the Cosine-Haversine Formula. When tables of natural and logarithmic haversines are available, calculation of the quantity $n$ is simpler than it is by the Law of Cosines.

*Example 2.* Given: $A$ (40° N, 80° W), $B$ (30° N, 50° W).
Find: distance $AB$, i.e., $n$, in nautical miles.

Formula: $\text{hav } n = \text{hav } (d_1 \sim d_2) + \cos d_1 \cos d_2 \text{ hav } N$

Here $d_1 = 40°$ and $d_2 = 30°$. $\therefore$ $(d_1 \sim d_2) = 10°$.

Also, $N = 80° - 50° = 30°$.

$\therefore$ $\text{hav } n = \text{hav } 10° + \cos 40° \cos 30° \text{ hav } 30°$

$$\begin{aligned} \text{hav } 10° &= 0.00760 \\ x &= \underline{0.04444} \\ \text{hav } n &= 0.05204 \end{aligned} \qquad \begin{aligned} \log \cos 40° &= 9.88425 \\ \log \cos 30° &= 9.93753 \\ \log \text{hav } 30° &= \underline{8.82599} \\ \log x &= 8.64777 \\ x &= 0.04444 \end{aligned}$$

$= 26° 22' = 1582$ nautical miles.

In the exercises of Group Forty-two use this formula to compute the spherical distance between the two points given in each instance. Compute arcs to the nearest minute and final distances in nautical miles to four significant figures. A partial table of haversines, both logarithmic and natural, accompanies these exercises. If your own logarithmic and trigonometric tables are four-place tables, round off the values given in the haversine table to four figures in doing your computation. Since these tables are computed here for the given angle read to the nearest minute only, the corresponding values of the log hav and nat hav can vary some from the values which are tabulated.

## EXERCISES

*Group Forty-two*

**1.** $A$ (47° N, 60° W)
$B$ (65° N, 15° W)

**2.** $A$ (60° N, 165° W)
$B$ (15° N, 90° W)

**3.** $A$ (35° N, 78° W)
$B$ (30° S, 50° W)

**4.** $A$ (45° N, 64° W)
$B$ (30° S, 30° E)

**5.** *$A$ (35° N, 118° W)
$B$ (10° S, 150° E)

**6.** New York (40° 28′ N, 74° W)
Liverpool (53° 24′ N, 3° 4′ W)

**7.** Boston (42° 23′ N, 71° 4′ W)
Capetown (33° 56′ S, 18° 29′ E)

**8.** Chicago (41° 50′ N, 87° 35′ W)
San Juan (18° 28′ N, 66° 7′ W)

**9.** Seattle (47° 36′ N, 122° 20′ W)
Montevideo (34° 53′ S, 56° 16′ W)

**10.** Nome (64° 30′ N, 165° 24′ W)
Manila (14° 34′ N, 120° 57′ E)

**Haversine Table**

| Angle | Log hav | Nat hav |
|---|---|---|
| 10° | 7.88059 − 10 | 0.00760 |
| 12° 56′ | 8.10327 | 0.01268 |
| 18° | 8.38867 | 0.02447 |
| 21° 28′ | 8.54014 | 0.03468 |
| 23° 22′ | 8.61286 | 0.04101 |
| 26° 22′ | 8.71634 | 0.05204 |
| 28° | 8.76735 | 0.05853 |
| 29° 40′ | 8.81637 | 0.06552 |
| 29° 56′ | 8.82400 | 0.06668 |
| 30° | 8.82599 | 0.06699 |
| 45° | 9.16568 | 0.14645 |
| 48° | 9.21851 | 0.16539 |
| 49° 56′ | 9.25081 | 0.17816 |
| 65° | 9.46043 | 0.28869 |
| 66° 4′ | 9.47300 | 0.29716 |
| 69° 34′ | 9.51246 | 0.32543 |
| 69° 35′ | 9.51255 | 0.32550 |
| 69° 52′ | 9.51568 | 0.32785 |
| 70° 9′ | 9.51879 | 0.33021 |
| 70° 56′ | 9.52720 | 0.33667 |
| 73° 39′ | 9.55539 | 0.35925 |
| 75° | 9.56889 | 0.37059 |
| 76° 19′ | 9.58175 | 0.38172 |
| 82° 29′ | 9.63808 | 0.43459 |
| 89° 33′ | 9.69555 | 0.49607 |
| 92° | 9.71387 | 0.51745 |
| 94° | 9.72825 | 0.53488 |
| 97° 21′ | 9.75119<br>9.75128 | 0.56388<br>0.56400 |
| 101° 25′ | 9.77740 | 0.59896 |
| 111° 49′ | 9.83616 | 0.68574 |
| 113° 21′ | 9.84394<br>9.84398 | 0.69813<br>0.69820 |

(An excellent table of haversines for angles from 0° to 180° may be found in Bowditch: *American Practical Navigator*, or in any standard manual of navigation).

* Take care in computing angle $N$ of the formula.

**249. Sailing the Great Circle Route in Actual Practice.** The great circle route, being the shortest route between two given points (§ 191), would seem to be the ideal one, especially if the voyage to be made is a long one. However, any great circle (except a meridian or the equator) crosses successive meridians at a constantly changing angle. Thus, if a ship were to sail *exactly* along a great circle, its course would have to be in a state of *continuous change.* Since the problem of navigating a ship in this manner is so involved as to be practically out of the question, an approximation to a great circle route is adopted when a more or less direct route is demanded.

This approximation to a true great circle route is achieved by means of a succession of short curves or arcs known as *rhumb lines.* A *rhumb line* or *loxodrome* is a curve on the Earth's surface which cuts all meridians at a constant angle. (If a rhumb line is extended sufficiently it forms a spiral which draws ever nearer to a pole.) Thus, a ship sailing along a rhumb line is enabled to maintain a fixed course. In approximating the true great circle route by means

Fig. 280

of these successive rhumb lines the ship may, therefore, be kept on one fixed course after another, — the length of each of the rhumb lines sailed being small or great depending upon the desired degree of approximation to the true great circle. Approximating the true great circle route from $A$ to $B$ is analogous to moving in a plane from a point $A$ to a point $B$ along a series of arcs whose extremities lie on the straight line-segment $AB$ (Fig. 280), — the shorter the separate arcs, the closer being the approximation to the direct path from $A$ to $B$. Obviously, a ship's route may be laid along a meridian or along the equator or along a parallel of latitude which is very near the equator; and the problem of navigating such a route is not difficult since the ship's course in effect may be kept constant. But in general the term "sailing the great circle route" as it is employed in practice implies sailing along a succession of rhumb lines which approximate the true great circle path.

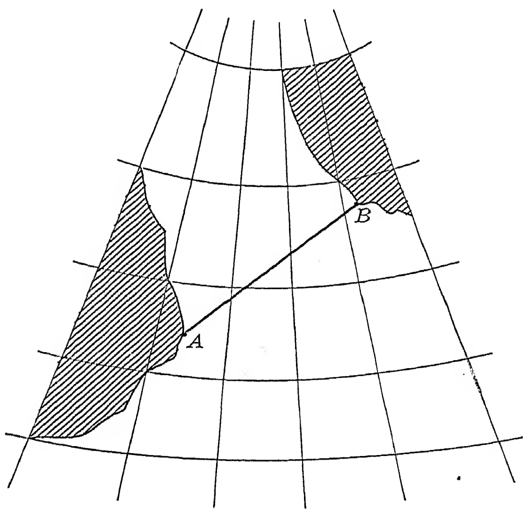



Fig. 281

In laying out the great circle route just discussed the navigator resorts to two types of maps or charts: the *great circle chart* and the *Mercator's chart*.

A great circle chart (Fig. 281) depicts a portion of the Earth's surface distorted in such a way that all great circle arcs appear as straight lines. Thus, in Fig. 281, the straight line-segment $AB$ represents the great circle arc connecting the corresponding points $A$ and $B$ on the globe.

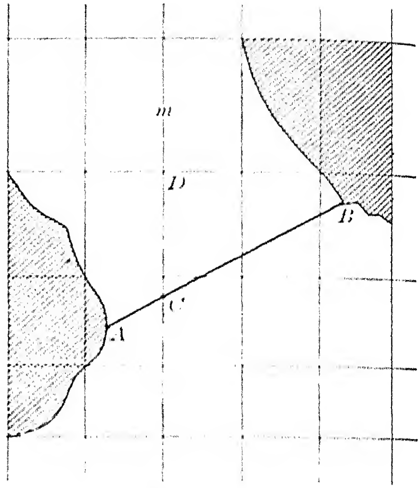



Fig. 282

On the Mercator's chart (Fig. 282) the Earth's surface is again pictured in distortion but this time of such kind that all rhumb lines appear as straight lines. In Fig. 282, which is a Mercator's chart supposedly of the same region as that pictured in Fig. 281, straight line-segment $AB$ represents the rhumb line connecting the corresponding points $A$ and $B$ on the globe. Equally spaced meridians of the Earth appear as equally spaced straight lines which are parallel. Parallels of latitude appear as parallel straight lines which are perpendicular to the meridian lines. Note that on the chart the distances between equally spaced parallels of latitude become greater and greater as the parallels recede from the equator. It is important to note that the Mercator's chart reproduces all angles of the Earth's surface faithfully. For example, in Fig. 282, the angle $DCB$ formed by the line $AB$ and the meridian $m$ is exactly equal to the angle between the two corresponding curves on the globe.

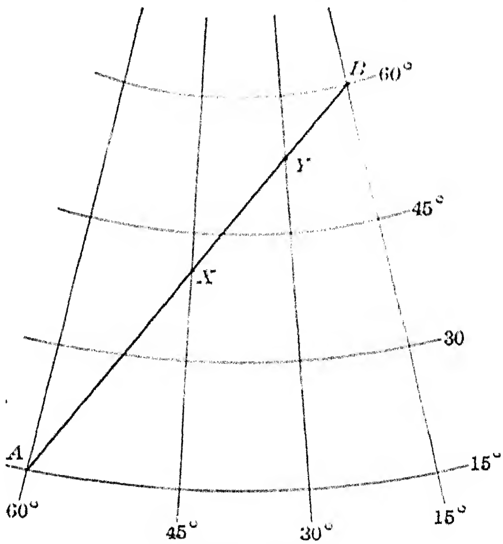



Fig. 283

*Example 3.* Lay out a great circle route from $A$ (15° N, 60° W) to $B$ (60° N, 15° W). (Use Figs. 283, 284.)

1) On the great circle chart (Fig. 283) lay a straight edge across points $A$ and $B$. If this line is seen to pass clear of any islands or shoals

indicated on the chart then a great circle route may be used. Draw the straight line $AB$, remembering that this line represents the great circle arc on the Earth.

2) Note the coördinates (lat. and long.) of several conveniently situated points on $AB$ such as $X$ and $Y$. Mark these points. On a genuine chart each "square" bounded by two parallels and two meridians is subdivided into smaller "squares" by dotted lines so spaced that any point may be plotted accurately, and the coördinates of any point already plotted on the chart can be read accurately.
3) Now plot the points $A$, $X$, $Y$, $B$ on the Mercator's chart (Fig. 284) which in practice is also ruled for accurate plotting and reading.
4) In Fig. 284 draw the chords $AX$, $XY$, $YB$. Each of these chords must represent a rhumb line. These three chords taken in succession represent the required route. Here, for sake of illustration, only three chords were used. Obviously any greater number of chords might have been used to approximate more closely the true great circle if in step 2 a greater number of points $X$, $Y$, etc. had been tabulated.

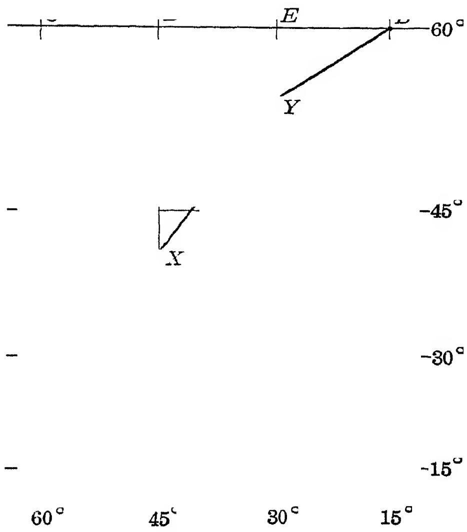


FIG. 284

5) The angles $CAX$, $DXY$, $EYB$ give correctly the courses to be taken at $A$, $X$, $Y$, respectively. Moreover, the positions (lat. and long.) of $A$, $X$, $Y$, $B$ being known, the route is completely determined. On a genuine large-scale Mercator's chart these angles and the lengths of the chords (nautical miles) may be read fairly accurately. There are, of course, mathematical means of calculating the angles and the lengths of the chords. These methods will be discussed after Ex. 6 in the following group of exercises.

## EXERCISES

### *Group Forty-three*

1. Procure a great circle chart of the North Atlantic and the corresponding Mercator's chart, and practice laying out great circle routes, as for example:

(a) New York to Plymouth, England,
(b) Norfolk, Virginia to Brest, France,
(c) Halifax, Nova Scotia to Lisbon, Portugal,
(d) San Juan, Puerto Rico to the Azores.

2. In Example 3 just discussed find by Spherical Trigonometry the true great circle distance from $A$ to $B$.

3. In Example 3 find by Spherical Trigonometry the latitudes of points $X$ and $Y$, respectively.

**4.** A great circle chart is made as follows: the globe (Earth) is placed tangent to a plane $M$ at a point $T$ (Fig. 285). Points of the globe's surface are then projected upon $M$ by means of rays from the center of the sphere. Prove by Solid Geometry that the projection upon $M$ of any great circle arc of the sphere must be a straight line-segment.

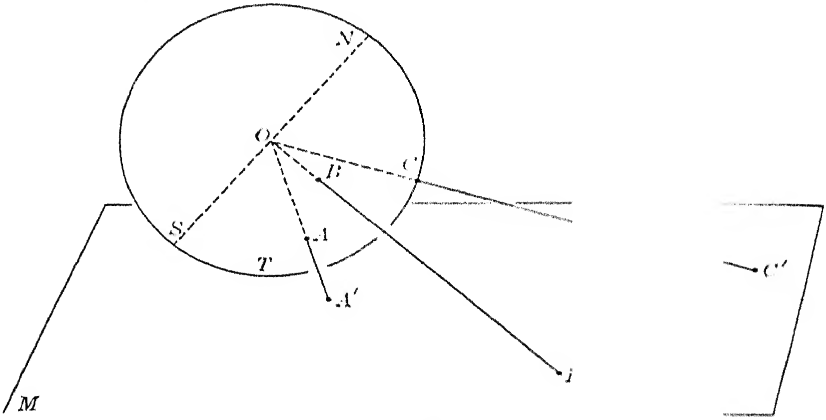



FIG. 285

**5.** In Ex. 4 describe the appearance of the projections upon $M$ of the Earth's meridians when $T$ is taken: (a) on the equator; (b) at either pole; (c) neither at a pole nor on the equator.

**6.** How much of the Earth's surface can be depicted on a great circle chart at once: half, more than half, or less than half?

On the Mercator's chart the actual number of degrees in angle $NAB$ is the course of a ship sailing from $A$ along the rhumb line $AB$ (Fig. 286). Therefore, the course ($\angle NAB$ or $\angle ABC$) can be found by solving the right triangle $ACB$ by Plane Trigonometry, — provided that the lengths $AC$ and $CB$ can be expressed in the same units.

The unit generally chosen for this purpose is a segment equal to the length of one minute of arc of the equator as the equator appears on the Mercator's chart. Since, on this chart, a one-minute segment of the equator equals in length a one-minute segment of any parallel of latitude, the number of these units in $AC$ of Fig. 286 must be the same as the number of minutes in the difference in longitude of $A$ and $C$ or $A$ and $B$. Expressed in these units the length of $AC$ is therefore $(75 - 30)60 = 2700$. These units are called *meridional parts*. In order to express the length of $CB$ in the same units refer to a table of

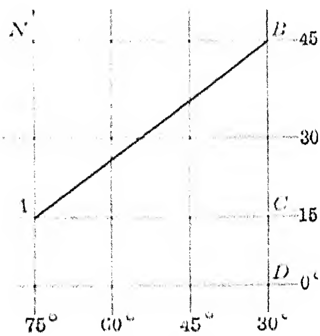



FIG. 286

meridional parts in which the latitude of any chosen point is tabulated in terms of meridional parts. From the table the number of meridional parts, M.P., for $DB$ (45°) is 3013; M.P. for $DC$ (15°) is 904. Therefore, M.P. for $CB = 3013 - 904 = 2109$. Hence, $\tan ABC = \frac{AC}{CB} = \frac{2700}{2109}$; and $\angle ABC = 52° \, 1'$.

To find the length of $AB$ in nautical miles it is necessary first to express the length of $CB$ in nautical miles. The number of nautical miles in $CB$ is the same as the number of minutes in the difference in latitude, or $(45 - 15)60$ or $30(60)$ or 1800.

Therefore, $$AB = \frac{1800}{\cos 52° \, 1'} = 2925 \text{ nautical miles.}$$

The ship's course is 52° 1′, and the length of the rhumb line route $AB$ is 2925 nautical miles.

In Exs. 7–15 calculate the starting course (at $A$) of a ship sailing the rhumb line $AB$. Find the length of the rhumb line in nautical miles. Use the partial table of meridional parts given at the right.

**7.** $A$ (50° N, 45° W)
$B$ (40° N, 28° W)

**8.** $A$ (33° S, 56° W)
$B$ (55° S, 0°)

**9.** $A$ (21° N, 155° W)
$B$ (38° N, 124° W)

**10.** $A$ (25° S, 128° W)
$B$ (45° S, 75° W)

**11.** $A$ (34° N, 116° W)
$B$ (15° S, 135° W)

**12.** $A$ (10° S, 150° E)
$B$ (18° N, 167° E)

**13.** $A$ (50° N, 157° E)
$B$ (15° S, 90° W)

**14.** $A$ (53° N, 180°)
$B$ (38° N, 124° W)

**15.** $A$ (33° N, 81° W)
$B$ (48° N, 4° W)

| *Latitude* | *Meridional Parts* |
|---|---|
| 10° | 599 |
| 15° | 904 |
| 18° | 1091 |
| 21° | 1281 |
| 25° | 1540 |
| 33° | 2087 |
| 34° | 2158 |
| 38° | 2454 |
| 40° | 2608 |
| 45° | 3013 |
| 48° | 3274 |
| 50° | 3457 |
| 53° | 3745 |
| 55° | 3949 |

**16.** In Ex. 7 find by Spherical Trigonometry the great circle distance from $A$ to $B$. Compare with the rhumb line distance.

**17.** Do the same in Ex. 15.

**250. The Celestial Sphere.** Fundamental to the study of Nautical Astronomy is the concept *celestial sphere*. The celestial sphere is an imaginary hollow sphere of gigantic size which envelops the Earth and is concentric with it. So great

is the radius of this imagined sphere that by comparison the length of the Earth's radius is negligible. Nautical Astronomy is not concerned with the actual size of the celestial sphere but rather with points lying on its surface and the solutions (in angular measure) of certain spherical triangles determined by these points. All the heavenly bodies: sun, moon, stars and planets visible from the Earth, are imagined to lie on the surface of the celestial sphere and to move across its surface. One of these heavenly bodies, conveniently selected, is always taken as one vertex of any one of the spherical triangles just mentioned. Figure 287 depicts the celestial sphere with the Earth inside it. Refer to this diagram while studying the following discussion.

*Description of the Celestial Sphere.* Three points on the celestial sphere which are of especial interest are: (a) the *poles* (either $P$ or $Q$), (b) some conveniently chosen *celestial body* $M$ such as the sun, (c) a point $Z$ called the *zenith*. The celestial poles $P$ and $Q$ are determined by extending the Earth's axis north and south, respectively, to meet the surface of the celestial sphere. $P$ is the *celestial north pole*, $Q$ the *celestial south pole*; $PQ$ is the *celestial axis*. The *body* $M$ may be the sun, moon, or some one of the stars or planets familiar to navigators. The point $G$ which is directly under $M$ on the Earth's surface is called the *geographic position* (GP) of $M$. Geometrically, points $M$, $G$, $O$ (center of the Earth) are collinear. If $A$ is some given terrestrial point, the *zenith* of $A$ is the celestial point $Z$ which appears directly over $A$. Points $Z$, $A$, $O$ are collinear.

The positions of celestial bodies and points may be expressed by a celestial coördinate system virtually corresponding to the latitude-longitude system on the Earth's surface. The basic lines of reference in one such coördinate system are the *celestial equator* and the *celestial prime meridian*, each of which in turn is determined by extending the plane of the terrestrial equator and the plane of the terrestrial prime meridian to cut the surface of the celestial sphere. In Fig. 287, $\overparen{D'JC''T'}$ is the celestial equator, and $\overparen{PT'LQ}$ is the celestial prime meridian. The general term *celestial meridian* is used in the same sense as the term meridian on the Earth's surface. In particular, the celestial meridian which passes through the body $M$ is called the *hour circle* ($\overparen{PMFD'Q}$ in Fig. 287). The sph $\angle t$ formed by the hour circle and the celestial meridian through $Z$ is called the *local hour angle* (LHA) of $M$. Sph $\angle \phi$ formed by the hour circle and the celestial prime meridian is the *Greenwich hour angle* (GHA) of $M$. Similarly, sph $\angle \theta$ may be regarded as the GHA of $Z$. GHA for any celestial body or point is measured positively westward up to 360° starting at the celestial prime meridian. In this respect GHA differs from terrestrial longitude which is measured 180° west and 180° east from the prime meridian. The *declination* (Del.) of a celestial point is its angular distance north or south of the celestial equator. Thus, Del. of $M$ is $\overparen{D'M}$; Del. of $Z$ is $\overparen{C''Z}$. Note that Del. of $M$ = lat. of $G$, Del. of $Z$ = lat. of $A$.

Del. and GHA are the two coördinates of a celestial point in the coördinate system just outlined. Del. and GHA of those celestial bodies which are im-

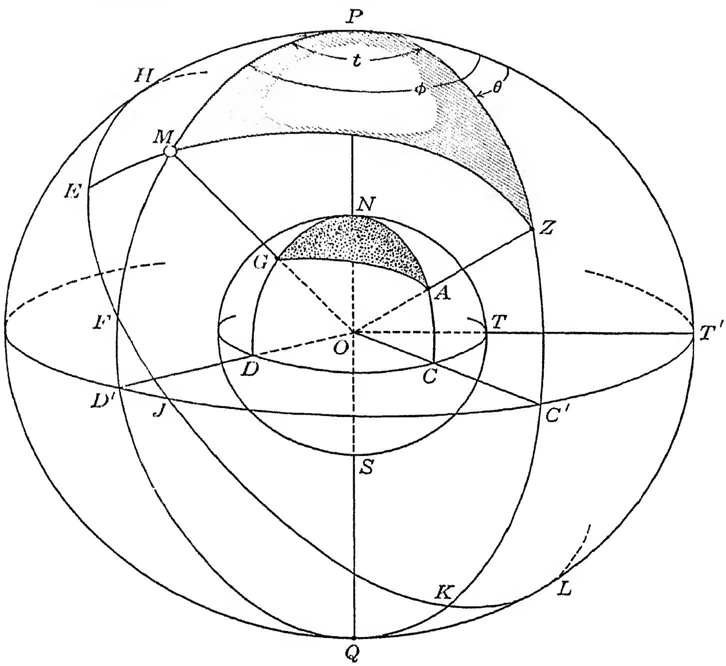



Fig. 287

$P$: celestial north pole
$Q$: celestial south pole
$PQ$: celestial axis

$M$: celestial body
$G$: GP of $M$

$A$: a given terrestrial point
$Z$: zenith of $A$

$\widehat{DCT}$: terrestrial equator
$\widehat{D'JC'T'}$: celestial equator

$\widehat{NTS}$: terrestrial prime meridian
$\widehat{PT'LQ}$: celestial prime meridian

$\widehat{PMFD'Q}$: hour circle of body

$\widehat{PZC'KQ}$: zenith meridian

sph $\angle t$: LHA of $M$
sph $\angle \phi$: GHA of $M$
sph $\angle \theta$: GHA of $Z$

$\left.\begin{array}{r}\widehat{D'M}\text{: Dcl. of } M\\ \widehat{DG}\text{: lat. of } G\end{array}\right\}$ Dcl. $M$ = lat. $G$

$\left.\begin{array}{r}\text{sph } \triangle PMZ\\ \text{sph } \triangle NGA\end{array}\right\}$ : the astronomical triangle

$\widehat{HEFJKL}$: celestial horizon (has $Z$ for pole)

$\widehat{EM}$: altitude of $M$ ($\widehat{EM}$ never exceeds 90°)

$\widehat{ZM}$: zenith distance of $M$ (alt. $M$ + zenith dist. $M$ = 90°)

sph $\angle PZM$: azimuth of $M$

$\widehat{PM}$: polar distance of $M$

portant to navigators are listed in the Nautical Almanac periodically through each day of the year, the time of day corresponding to each tabulation being given in Greenwich Civil Time. In the Nautical Almanac a Del. north is given with a + sign, a Del. south with a − sign.

Anything approaching a complete treatment of "time" or "time of day" is far too involved to be given here. We shall mention only one method of reckoning time known as *true sun time* or *apparent time*. When the hour circle of the sun coincides with the zenith meridian of a given terrestrial point $A$ the apparent time at $A$ is said to be 12 : 00 noon. In reality the Earth rotates about its own axis in an easterly direction. Therefore, from the viewpoint of the navigator the Earth appears to remain stationary while the sun moves across the surface of the celestial sphere in a westerly direction about the Earth. If we assume that the sun makes such a circuit once in exactly 24 hours, then an increase of 15° in the sun's GHA must correspond to an elapse of 1 hour of time since $360 \div 24 = 15$. Thus, in Fig. 287, calling $M$ the sun, if $t$ is 15° with $M$ west of $Z$, the apparent time at $A$ is 1 hour after noon or 1 : 00 P.M. If $t = 30°$ with $M$ east of $Z$ the apparent time at $A$ is 2 hours before noon or 10:00 A.M. The apparent time at $A$, therefore, is known if the number of degrees in $t$ is known, $t/15$ being the number of hours before or after noon accordingly as $M$ is east or west of $Z$.

In Fig. 287 sph $\triangle PMZ$ having as sides the meridian arcs $PZ$ and $PM$ and the great circle arc $ZM$ is called the *astronomical triangle*. By solving this triangle when three of its parts are available a navigator may determine his ship's latitude and longitude. $\widehat{PM}$ is called the *polar distance* of the body $M$, a term which must not be confused with the definition of polar distance given in § 188. $\widehat{ZM}$ is the *zenith distance* of $M$. Sph $\angle PZM$ is the *azimuth* of $M$. It is important to note that $\triangle PMZ$ is exactly the same as the $\triangle NGA$ on the Earth, angular measure being implied. One triangle is really the projection of the other. Either of these triangles may be thought of as the astronomical triangle. Recall that, in effect, it was $\triangle NGA$ which was used in §§ 247, 248 to solve problems involving terrestrial great circle distances, positions, bearings, etc.

The great circle $HEFJKL$ which has $Z$ for a pole is known as the *celestial horizon* with reference to $A$. Celestial horizon must not be confused with the "visible horizon" which is the apparent line of separation of Earth from sky as seen from aboard ship. The plane of the celestial horizon passes through $O$, while that of the visible horizon is parallel to the celestial horizon being the plane of a small circle of the Earth. In Fig. 288 $V$ is the eye of an observer at $A$, $Z$ is the zenith, $M$ is a celestial body. $C$ and $D$ are points of contact of tangents to the Earth from $V$, and $CD$ represents the plane of the visible horizon. $EF$ is the plane of the celestial horizon. The *altitude* of the body $M$ is its angular distance above $EF$, viz. $\angle \delta$ or $\widehat{EM}$. *Altitude* of $M$ is its spherical distance from the celestial horizon. The altitude of $M$ can be obtained by direct observation with a sextant. The angle $\psi$, the angle of elevation (virtually) of $M$ from $V$, is noted. A suitable correction from tables is then applied to the reading to

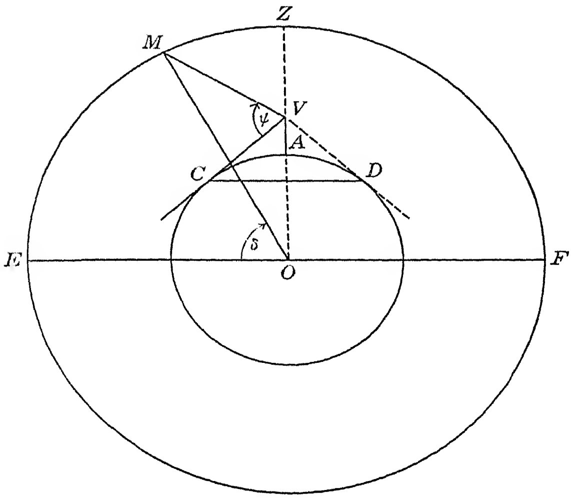



FIG. 288

obtain the number of degrees in the required $\angle\,\delta$ or $\widehat{EM}$. Returning now to Fig. 287 we note that $\widehat{ZE}$ must be a quadrant since $Z$ is a pole of $HEFJKL$. $\widehat{EM}$ is the altitude of $M$. Therefore, for any visible body $M$: altitude + zenith distance = 90° always.

## EXERCISES

### *Group Forty-four*

**1.** In Fig. 287 show that the angle $\theta$ equals the longitude of $A$ if $\theta$ does not exceed 180°, and that $360° - \theta$ equals the longitude of $A$ if $\theta$ does exceed 180°.

**2.** Find the longitude of a point $A$ if the GHA of its zenith is 70°; 130°; 180°; 260°; 320°.

**3.** Find the GHA of the zenith of $A$ if long. $A$ = 30° W; 40° E; 180°.

**4.** The longitude of $A$ is 75° W. Find the apparent time at $A$ at the instant when the apparent time at Greenwich is 8 : 00 A.M.; 12 : 00 noon; 6 : 00 P.M.; 12 : 00 midnight.

**5.** Do Ex. 4 assuming that long. $A$ = 135° E.

**6.** When it is 8 : 00 A.M. at New York (long. 74° W) what is the time at Greenwich?

**7.** In the afternoon the LHA of the sun relative to a given point $A$ is 22° 30′. Find the apparent time at $A$.

**8.** In the forenoon the LHA of the sun is found to be 55°. What is the apparent time at the given point?

**9.** At the same instant the apparent time at $A$ is 10:00 A.M. and that at Greenwich is 1 : 00 P.M. Find the GHA of $A$'s zenith and hence find the longitude of $A$.

**10.** Find the longitude of a point $B$ if at the same instant the apparent time at $B$ is 3 : 00 P.M. and that at Greenwich is 6 : 00 A.M.

**11.** When the apparent time at Greenwich is 12 : 00 midnight what is the longitude of a point $A$ at which the apparent time is 12 : 00 noon?

**12.** If $A$ is in the western hemisphere, and if $T$ represents the number of hours before or after apparent noon at $A$, show that:
(a) long. $A$ = ($15T$ + GHA of sun) if sun is east of $A$;
(b) long. $A$ = (GHA of sun − $15T$) if sun is west of $A$.

**13.** If $A$ is in the eastern hemisphere, show that
(a) long. $A$ = (360 − $15T$ − GHA sun) if sun is east of $A$;
(b) long. $A$ = (360 − GHA sun + $15T$) if sun is west of $A$.

**14.** Find the zenith distance of $M$ if the altitude of $M$ is 35°.

**15.** Find the polar distance of $M$ if Del. $M$ is 20° N.

**16.** Find $PZ$ of the astronomical triangle if lat. $A$ = 40° N.

If the *latitude* of $A$ is known together with the *declination* and *altitude* of the sun, then all three sides of the astronomical triangle $PZM$ are readily found. Angle $t$ may then be found by Spherical Trigonometry, and hence the *apparent time at* $A$, either forenoon or afternoon, can be found.

**17.** Find the apparent time at $A$ from the data: lat. $A$ = 40° N, Del. $M$ = 20° N, alt $M$ = 35°, observation being made in the afternoon.

In Exs. 18–21 find the apparent time at the places indicated.

| | *Lat. of place* | *Del. M* | *Alt. M* | *A.M. or P.M.* |
|---|---|---|---|---|
| **18.** | Santiago (19° 57′ N) | 22° N | 30° | P.M. |
| **19.** | Dublin (53° 23′ N) | 15° N | 40° | A.M. |
| **20.** | Chicago (41° 51′ N) | 20° N | 50° | P.M. |
| **21.** | Melbourne (37° 50′ S) | 18° N | 20° | A.M. |

**22.** Referring to Fig. 287 derive the formula:

$$\boxed{\text{hav } t = \csc p \sec d \sin s \cos (s - h)}$$

where $t$ = LHA of $M$, $p$ = polar distance of $M$, $d$ = latitude of $A$, $h$ = altitude of $M$, $s = \frac{1}{2}(p + d + h)$. Use Law of Cosines, definition of haversine (§ 248), T 5–b, 6–a, 9–b. This formula can be used to advantage in problems such as Exs. 17–21 preceding.

**23.** Do Ex. 18 by this formula.

**24.** Do Ex. 20 by this formula.

The *apparent time of sunrise or sunset* at a given point $A$ can be found easily if the *latitude of* $A$ and the *declination of the sun* are known. At sunrise or sunset the altitude of the sun is assumed to be 0°. Therefore, the zenith distance $ZM$ will be 90°. Sph $\triangle PZM$ is then seen to be a quadrantal triangle which can be solved for angle $t$ by the methods of § 242, or by the Law of Cosines, or by the formula of Ex. 22.

In Exs. 25–28 find the apparent time of sunrise or sunset as indicated.

**25.** Sunrise at Portland, Me. (43° 39′ N), Dcl. sun = 18° 11′ S.

**26.** Sunset at New Orleans (29° 58′ N), Dcl. sun = 23° 2′ S.

**27.** Sunrise at Rio de Janeiro (22° 54′ S), Dcl. sun = 20° 40′ N.

**28.** Sunset at San Luis Obispo, Cal. (35° 10′ N), Dcl. sun = 23° 9′ N.

The latitude of a point $A$ is readily determined by a noon observation of the sun, the sun's declination being known. At 12:00 noon, apparent time, the sun $M$ appears on the zenith meridian of $A$. Hence, by direct observation the zenith distance $ZM$ can be determined.

**29.** In Fig. 287 visualize $M$ moved over to the zenith meridian. Representing the zenith distance $ZM$ by $z$, show that:

(a) Lat. $A$ = Dcl. $M + z$, if lat. $A >$ Dcl. $M$ and if lat. $A$ and Dcl. $M$ are of the same name. (Latitudes or declinations are said to be of the *same name* if both are measured north or both are measured south. If one is measured north and the other south, they are said to be of *different name*.)

(b) Lat. $A$ = Dcl. $M - z$, if lat. $A <$ Dcl. $M$ and if lat. $A$ and Dcl. $M$ are of the same name.

(c) Lat. $A = z -$ Dcl. $M$, if lat. $A$ and Dcl. $M$ are of different name.

**30.** Find the latitude of a point $A$ by noon observation of sun from the following data:

(a) $z = 20° 10'$, Dcl. = 18° 5′ N, lat. > Dcl. and same name;
(b) $z = 6° 45'$, Dcl. = 23° 12′ N, lat. < Dcl. and same name;
(c) $z = 63° 10'$, Dcl. = 22° S, lat. and Dcl. of different name;
(d) $z = 18° 30'$, Dcl. = 15° S, lat. > Dcl. and same name;
(e) $z = 12°$, Dcl. = 22° 45′ S, lat. < Dcl. and same name.

When the observed body $M$ is not on the celestial meridian of the observer, i.e., not on the zenith meridian, the *latitude of A* can be found from the astronomical triangle $PZM$ provided that the *altitude*, *declination*, and LHA of $M$ are known. If the three latter quantities are known then $\widehat{ZM}$, $\widehat{PM}$, $t$ of $\triangle PZM$ are at once determined. By Spherical Trigonometry $\widehat{PZ}$ can then be found. If $A$ is north of the equator, lat. $A = \widehat{C'Z} = 90° - \widehat{PZ}$; if $A$ is south of the equator, lat. $A = \widehat{PZ} - 90°$. In either case the latitude of $A$ is known if $\widehat{PZ}$ is first found. A method of calculating $\widehat{PZ}$ is illustrated by the following example.

Given: $A$ north of the equator; for a certain star $M$: Dcl. = 60° N, alt. = 40°, LHA = 45°.

Find the latitude of $A$.

Since Dcl. $M$ = 60° N, $\widehat{PM} = 30°$ (cf. Fig. 287). Since alt. = 40°, $\widehat{ZM} = 50°$. LHA = $t$ = 45°.

From $M$ draw altitude $\widehat{MW}$ in sph $\triangle PZM$ (Fig. 289). Applying Napier's Rule to rt $\triangle PWM$:

$$\widehat{PW} = 22° 12'. \quad \text{Also,} \quad \widehat{MW} = 20° 43'.$$

Again, in rt $\triangle ZWM$: $\widehat{WZ} = 46° 35'$.

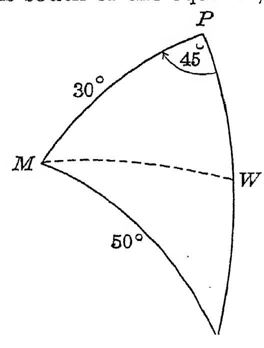



FIG. 289

$\therefore \quad \widehat{PZ} = \widehat{PW} + \widehat{WZ} = 22°12' + 46°\,35' = 68°\,47'.$

$\therefore \quad \text{lat. } A = 90° - \widehat{PZ} = 90° - 68°\,47' = 21°\,13'\text{ N}.$

In Exs. 31–34 find the latitude of $A$ from the accompanying data.

| | *Hemisphere* | *Del. M* | *Alt. M* | *LHA of M* |
|---|---|---|---|---|
| **31.** | north | 60° N | 20° | 56° |
| **32.** | north | 15° N | 22° | 72° |
| **33.** | south | 23° N | 30° | 42° |
| **34.** | south | 16° S | 25° | 60° |

**35.** A ship $B$ is in the South Pacific. The apparent time at $B$ is known to be 8 : 00 A.M., while that at Greenwich is 6 : 00 P.M. The sun's declination is 22° N, and from $B$ the sun's altitude is found to be 15°. Find the latitude and longitude of $B$.

# *Chapter Eighteen*

## SPHERICAL TRIGONOMETRY: SUPPLEMENTARY FORMULAS FOR OBLIQUE TRIANGLES

Each of the formulas offered in this chapter is with reference to an oblique spherical triangle $ABC$ with sides $a$, $b$, $c$ lettered in the customary fashion. The use of these formulas in solving an oblique triangle affords a refinement of procedure in comparison with the methods already presented, — especially if a complete solution of the triangle is required. It will be noted that without exception each of these new formulas is better adapted to logarithmic computation than is the Law of Cosines.

**251. The Half-angle Formulas** (For the case "SSS" of § 246).

(18) $$\tan \frac{A}{2} = \frac{f}{\sin (s-a)}$$

(19) $$\tan \frac{B}{2} = \frac{f}{\sin (s-b)}$$

(20) $$\tan \frac{C}{2} = \frac{f}{\sin (s-c)}$$

where $s = \frac{1}{2}(a+b+c)$,

and $$f = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}$$

*Derivation of Formula* (18)

1) $$\tan^2\left(\frac{A}{2}\right) = \frac{1-\cos A}{1+\cos A}$$ (T 8–$c$)

2) $$\frac{1 - \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}}{1 + \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}}$$ ( (11) )

3) $$= \frac{(\sin b \sin c + \cos b \cos c) - \cos a}{(\sin b \sin c - \cos b \cos c) + \cos a}$$

4) $$= \frac{\cos (b - c) - \cos a}{-\cos (b + c) + \cos a} = \frac{\cos a - \cos (b - c)}{\cos (b + c) - \cos a} \qquad \text{(T 6-}b\text{)}$$

5) $$= \frac{\sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a - b + c)}{\sin \frac{1}{2}(a + b + c) \sin \frac{1}{2}(b + c - a)} \qquad \text{(T 9-}d\text{)}$$

6) Let $s = \frac{1}{2}(a + b + c)$.
Then $\frac{1}{2}(a + b - c) = (s - c)$; $\frac{1}{2}(a - b + c) = (s - b)$; $\frac{1}{2}(b + c - a) = (s - a)$.

7) Subst. in 5): $\tan^2\left(\frac{A}{2}\right) = \frac{\sin (s - c) \sin (s - b)}{\sin s \sin (s - a)}$

8) $\therefore \tan \frac{A}{2} = \sqrt{\frac{\sin (s - c) \sin (s - b)}{\sin s \sin (s - a)}}$

9) $$= \frac{\sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}}{\sin (s - a)}$$

In step 9, denote the radical in the numerator by $f$.

10) $\therefore \quad \tan \frac{A}{2} = \frac{f}{\sin (s - a)}$. Similarly for (19) and (20).

*Note:* Compare formulas (18), (19), (20) with those of T 16.

**252. The Half-side Formulas** (For the case "AAA" of § 246).

(21) $\tan \frac{a}{2} = F \cos (S - A)$

(22) $\tan \frac{b}{2} = F \cos (S - B)$

(23) $\tan \frac{c}{2} = F \cos (S - C)$

where $S = \frac{1}{2}(A + B + C)$,

and $F = \sqrt{\frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}}$

(The derivation is similar to that of (18). Set $\tan^2\left(\frac{a}{2}\right) = \frac{1 - \cos a}{1 + \cos a}$, and in place of $\cos a$ substitute $\frac{\cos A + \cos B \cos C}{\sin B \sin C}$ from (14).)

**253. Napier's Analogies I** (For cases "SAS, AAS, SSA" of § 246).

$$\text{㉔} \quad \tan \frac{A+B}{2} = \left( \frac{\cos \dfrac{a-b}{2}}{\cos \dfrac{a+b}{2}} \right) \cot \frac{C}{2}$$

$$\text{㉕} \quad \tan \frac{A-B}{2} = \left( \frac{\sin \dfrac{a-b}{2}}{\sin \dfrac{a+b}{2}} \right) \cot \frac{C}{2}$$

*Derivation of Formula ㉔*

1) $\tan \dfrac{A+B}{2} = \dfrac{\tan \dfrac{A}{2} + \tan \dfrac{B}{2}}{1 - \tan \dfrac{A}{2} \tan \dfrac{B}{2}}$ (T 6–$c$)

2) $= \dfrac{\dfrac{f}{\sin (s-a)} + \dfrac{f}{\sin (s-b)}}{1 - \dfrac{f^2}{\sin (s-a) \sin (s-b)}}$ (⑱, ⑲)

3) $= \dfrac{f[\sin (s-a) + \sin (s-b)]}{\sin (s-a) \sin (s-b) - f^2}$

4) $\tan \dfrac{A+B}{2} \tan \dfrac{C}{2} = \dfrac{f^2[\sin (s-a) + \sin (s-b)]}{\sin (s-c)[\sin (s-a) \sin (s-b) - f^2]}$

5) $= \dfrac{f^2[\sin (s-a) + \sin (s-b)]}{\sin (s-a) \sin (s-b) \sin (s-c) - f^2 \sin (s-c)}$

6) $= \dfrac{f^2[\sin (s-a) + \sin (s-b)]}{f^2 \sin s - f^2 \sin (s-c)}$ (See derivation of ⑱, steps 9 and 10, for the equivalent of $f$.)

7) $= \dfrac{\sin (s-a) + \sin (s-b)}{\sin s - \sin (s-c)}$

8) $= \dfrac{\sin \dfrac{2s-a-b}{2} \cos \dfrac{a-b}{2}}{\sin \dfrac{c}{2} \cos \dfrac{2s-c}{2}}$ (T 9–a–b)

9) Let $2s = a + b + c$. $\therefore \dfrac{2s-a-b}{2} = \dfrac{c}{2}$ and $\dfrac{2s-c}{2} = \dfrac{a+b}{2}$

10) $\therefore \tan \dfrac{A+B}{2} \tan \dfrac{C}{2} = \dfrac{\sin \dfrac{c}{2} \cos \dfrac{a-b}{2}}{\sin \dfrac{c}{2} \cos \dfrac{a+b}{2}} = \dfrac{\cos \dfrac{a-b}{2}}{\cos \dfrac{a+b}{2}}$

11) $\therefore$ $\boxed{\tan \frac{A+B}{2} = \frac{\cos \frac{a-b}{2}}{\cos \frac{a+b}{2}} \cot \frac{C}{2}}$ Similarly for (25).

**254. Napier's Analogies II** (For cases "ASA, AAS, SSA" of § 246).

(26) $$\tan \frac{a+b}{2} = \frac{\cos \frac{A-B}{2}}{\cos \frac{A+B}{2}} \tan \frac{c}{2}$$

(27) $$\tan \frac{a-b}{2} = \frac{\sin \frac{A-B}{2}}{\sin \frac{A+B}{2}} \tan \frac{c}{2}$$

(The derivations of (26) and (27) correspond to those for § 253.)

**255. Law of Species for an Oblique Spherical Triangle.**

One-half the sum of any two sides must be of the same species as one-half the sum of the two angles opposite these sides.

*Proof*

1) In (26) the quantities $\cos \frac{A-B}{2}$ and $\tan \frac{c}{2}$ must each be positive, since $\frac{A-B}{2}$ and $\frac{c}{2}$ must each be acute.

2) $\therefore$ the quantity $\tan \frac{a+b}{2}$ will be positive or negative according as the quantity $\cos \frac{A+B}{2}$ is positive or negative.

3) That is, $\frac{a+b}{2}$ is greater or less than 90° according as $\frac{A+B}{2}$ is greater or less than 90°.

4) $\therefore$ $\frac{a+b}{2}$ and $\frac{A+B}{2}$ must always be of the same species.

*Use of the Formulas*

*Example 1.* Solve the $\Delta ABC$ in which $a = 41°$, $b = 52°$, $c = 45°$.
Use § 251. Here: $s = 69°$, $s - a = 28°$, $s - b = 17°$, $s - c = 24°$.

$$\tan\frac{A}{2} = \frac{f}{\sin(s-a)} = \frac{1}{\sin 28^\circ}\sqrt{\frac{\sin 28^\circ \sin 17^\circ \sin 24^\circ}{\sin 69^\circ}}$$

Use logs to find $\frac{A}{2}$ and hence find $A$. Find $B$ and $C$ similarly.

*Example 2.* Solve the $\triangle ABC$ in which $A = 63^\circ$, $B = 75^\circ$, $C = 80^\circ$. Use § 252. Here: $S = 109^\circ$, $S - A = 46^\circ$, $S - B = 34^\circ$, $S - C = 29^\circ$.

$$\tan\frac{a}{2} = F\cos(S-A) = \left(\sqrt{\frac{-\cos 109^\circ}{\cos 46^\circ \cos 34^\circ \cos 29^\circ}}\right)\cos 46^\circ$$

Use logs to find $\frac{a}{2}$, and hence find $a$. Find $b$ and $c$ similarly.

*Example 3.* Solve the $\triangle ABC$ in which $a = 142^\circ$, $b = 68^\circ$, $C = 147^\circ$.

1) Find $\frac{A+B}{2}$ from $\tan\frac{A+B}{2} = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}}\cot\frac{C}{2} = \frac{\cos 37^\circ}{\cos 105^\circ}\cot 73^\circ 30'.$

2) Find $\frac{A-B}{2}$ from $\tan\frac{A-B}{2} \quad \frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cot\frac{C}{2} = \frac{\sin 37^\circ}{\sin 105^\circ}\cot 73^\circ 30'.$

Solving: $\frac{A+B}{2} = 137^\circ 34'$

$\frac{A-B}{2} = 10^\circ 28'$

Adding: $A = 148^\circ 2'$
Subtracting: $B = 127^\circ 6'$

3) Find $c$ from $\tan\frac{a+b}{2} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}\tan\frac{c}{2}$, by solving for $\tan\frac{c}{2}$ and evaluating the result by logs.

*Example 4.* Solve the $\triangle ABC$ in which $C = 122^\circ$, $A = 30^\circ$, $b = 40^\circ$.

1) Find $\frac{c+a}{2}$ from $\tan\frac{c+a}{2} = \frac{\cos\frac{C-A}{2}}{\cos\frac{C+A}{2}}\tan\frac{b}{2} = \frac{\cos 46^\circ}{\cos 76^\circ}\tan 20^\circ.$

2) Find $\frac{c-a}{2}$ from $\tan\frac{c-a}{2} = \frac{\sin\frac{C-A}{2}}{\sin\frac{C+A}{2}}\tan\frac{b}{2} = \frac{\sin 46^\circ}{\sin 76^\circ}\tan 20^\circ.$

As in Example 3, find $c$ and $a$ separately by adding and then subtracting the values found for $\frac{c+a}{2}$ and $\frac{c-a}{2}$.

3) Find $B$ from $\tan \frac{C+A}{2} = \dfrac{\cos \frac{c-a}{2}}{\cos \frac{c+a}{2}} \cot \frac{B}{2}.$

*Example 5.* Solve the $\triangle ABC$ in which $a = 56°$, $b = 138°$, $B = 130°$. This is the case "SSA". Hence look for two solutions.

1) Find $A$ from $\sin A = \dfrac{\sin B \sin a}{\sin b} = \dfrac{\sin 130° \sin 56°}{\sin 138°}$

$$A = 71° 40' \quad \text{or} \quad 108° 20'.$$

By § 243 either of these values is acceptable. Hence there *are* two solutions to the triangle.

| Sol. I | Sol. II |
|---|---|
| $A = 71° 40'$ | $A = 108° 20'$ |

2) Find $C$ from § 253, using first one value of $A$ and then the other. The value of $C$ for which $A = 71° 40'$ must belong to Sol. I; the other value of $C$ belongs to Sol. II.
3) Find $c$ from § 254, using first one value of $A$ and then the other. The value of $c$ for which $A = 71° 40'$ must belong to Sol. I, and the other value of $c$ must belong to Sol. II.

## EXERCISES

### *Group Forty-five*

Use § 251 in Exs. 1–4 to find the local hour angle of the sun at a given point $A$ on the Earth (from Exs. 18–21 of Group 44).

**1.** Lat. $A = 19° 57'$ N, Del. $M = 22°$ N, alt. $M = 30°$

**2.** Lat. $A = 53° 23'$ N, Del. $M = 15°$ N, alt. $M = 40°$

**3.** Lat. $A = 41° 51'$ N, Del. $M = 20°$ N, alt. $M = 50°$

**4.** Lat. $A = 37° 50'$ S, Del. $M = 18°$ N, alt. $M = 20°$

**5.** Using § 251 find the azimuth of $M$ in each of the preceding exercises.

**6.** Given: $A$ (30° N, 80° W), $B$ (40° N, 50° W); spherical distance $AB$ is 1582 nautical miles. Find by § 251: (a) the true bearing of $B$ from $A$; (b) the true bearing of $A$ from $B$.

In Exs. 7–11 (from Exs. 1, 3, 4, 6, 9 of Group 41) use § 253 to find the starting course and finishing course of a ship sailing the true great circle route from $A$ to $B$, given:

**7.** $A$ (10° 30′ N, 20° 15′ W), $B$ (70° 25′ N, 65° W).

**8.** $A$ (37° 32′ N, 122° 13′ W), $B$ (21° 20′ N, 158° W).

**9.** $A$ (33° 43′ N, 118° 15′ W), $B$ (14° 34′ N, 120° 57′ E).

**10.** $A$ (29° 4′ S, 167° 58′ E), $B$ (34° 24′ N, 119° 43′ W).

**11.** $A$ (43° 4′ N, 70° 44′ W), $B$ (30° 20′ N, 30° 30′ W).

**12.** For a given point $A$ on the Earth: Dcl. $M = 24°$ N, lat. $A = 35°$ S, LHA of $M = 40°$. Use § 253 to find the azimuth of $M$.

In Exs. 13–19 use the formulas of this chapter to solve the triangles from the parts given.

**13.** $B = 30° 30'$, $C = 121° 25'$, $a = 41°$.

**14.** $a = 105° 10'$, $b = 50° 20'$, $c = 62° 28'$.

**15.** $A = 118° 5'$, $B = 128° 10'$, $C = 78° 40'$.

**16.** $b = 100°$, $c = 58°$, $A = 108°$.

**17.** $A = 21° 36'$, $B = 145° 12'$, $b = 135° 45'$.

**18.** $a = 100°$, $b = 65°$, $A = 98°$.

**19.** $a = 42°$, $b = 119°$, $A = 30°$.

**20.** Prove: $\tan \frac{A - B}{2} = \tan \frac{a - b}{2} \cot \frac{a + b}{2} \tan \frac{A + B}{2}$.

**21.** Prove: $\tan \frac{a - b}{2} = \tan \frac{A - B}{2} \cot \frac{A + B}{2} \tan \frac{a + b}{2}$.

**22.** Derive Formula (25).

**23.** Derive Formula (26).

**24.** Derive Formula (27).

**25.** Derive Formula (21) by applying (18) to the polar triangle of $\triangle ABC$.

**256. Spherical Radius of the Inscribed Circle. Geometric Significance of "$f$" in § 251.** If a circle of the sphere is tangent to the sides of a spherical triangle $ABC$, the circle is said to be *inscribed* in the triangle. (By "tangent" we mean here "touching each side in one point only".) In Fig. 290, the pole, $O$, of that circle is the *spherical incenter;* the polar distance, $r$, of the circle is the *spherical inradius.*

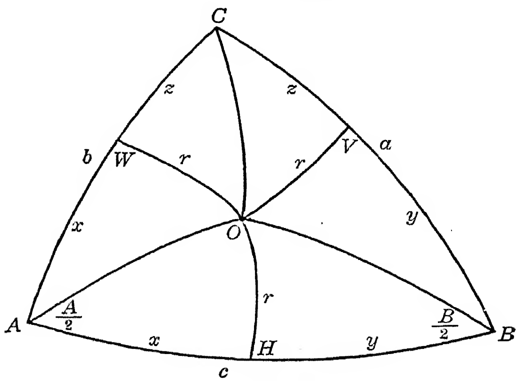



FIG. 290

1) Let $O$ be the pole (sph. incenter) of the inscribed circle. Let the points of contact be $H$, $V$, $W$, respectively.
2) Draw $\widehat{OA}$, $\widehat{OB}$, $\widehat{OC}$. It can be shown that these arcs bisect the angles, $A$, $B$, $C$ of $\triangle ABC$.
3) Draw $\widehat{OH}$, $\widehat{OV}$, $\widehat{OW}$. It can be shown that $\widehat{OH} = \widehat{OV} = \widehat{OW} = r$, the inradius. Moreover, $\widehat{OH}$, $\widehat{OV}$, $\widehat{OW}$ are each perpendicular to a side of the triangle.

4) In the rt $\triangle AHO$: $\sin x = \cot \frac{A}{2} \tan r$. (⑥)

5) $\therefore \tan \frac{A}{2} = \frac{\tan r}{\sin x}$

6) If $s$ represents $\frac{1}{2}(a + b + c)$, then $x = (s - a)$

7) $\therefore \tan \frac{A}{2} = \frac{\tan r}{\sin (s - a)}$

8) But $\tan \frac{A}{2} = \frac{f}{\sin (s - a)}$ (§ 251)

9) $\therefore \quad f = \tan r$

10) That is, $\tan r = \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}$

(Compare this formula with that of T 12 a.)

**257. Spherical Radius of the Circumscribed Circle. Geometric Significance of "$F$" in § 252.** If the vertices of a spherical triangle $ABC$ lie on the circumference of a circle of the sphere, this circle is said to be *circumscribed* about the triangle. The pole, $O$, of this circle is the *spherical circumcenter;* the polar distance, $R$, is the *spherical circumradius.*

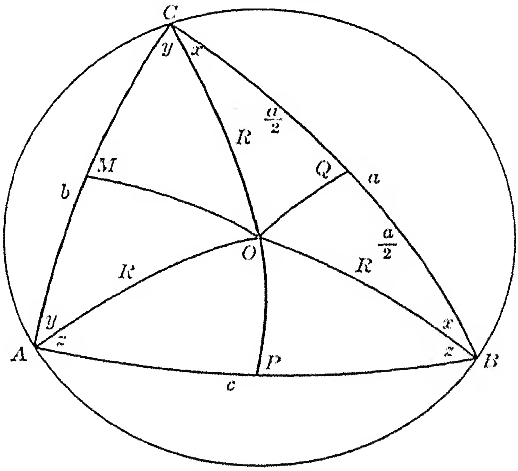



Fig. 291

1) Let $O$ be the pole (sph. circumcenter) of the circumscribed circle (Fig. 291).

2) Draw $\widehat{OP}$, $\widehat{OQ}$, $\widehat{OM}$ perpendicular to $\widehat{AB}$, $\widehat{BC}$, $\widehat{CA}$, respectively. It can be shown that $\widehat{OP}$, $\widehat{OQ}$, $\widehat{OM}$ bisect $\widehat{AB}$, $\widehat{BC}$, $\widehat{CA}$, respectively.

3) Draw $\widehat{OA}$, $\widehat{OB}$, $\widehat{OC}$. It can be shown that $\widehat{OA} = \widehat{OB} = \widehat{OC} = R$.

4) In the rt $\triangle OQB$: $\cos x = \tan \frac{a}{2} \cot R$ (②)

5) $\therefore \tan \frac{a}{2} = \tan R \cos x$

6) If $S$ represents $\frac{1}{2}(A + B + C)$, then $x = (S - A)$

7) $\therefore \tan \frac{a}{2} = \tan R \cos (S - A)$

8) But $\tan \frac{a}{2} = F \cos (S - A)$ (§ 252)

9) $\therefore$ $\boxed{F = \tan R}$

10) That is, $\tan R = \sqrt{\dfrac{- \cos S}{\cos (S - A) \cos (S - B) \cos (S - C)}}$

## PLANE TRIGONOMETRY: STANDARD FORMULAS

**1.**

| | $0°$ | $90°$ | $180°$ |
|---|---|---|---|
| sin | 0 | 1 | 0 |
| cos | 1 | 0 | $-1$ |
| tan | 0 | $\infty$ | 0 |
| cot | $\infty$ | 0 | $\infty$ |
| sec | 1 | $\infty$ | $-1$ |
| csc | $\infty$ | 1 | $\infty$ |

For any angle $A$ or $B$:

(a) $\sin A \csc A = 1$; (b) $\cos A \sec A = 1$; (c) $\tan A \cot A = 1$

**3.** (a) $\tan A = \dfrac{\sin A}{\cos A}$; (b) $\cot A = \dfrac{\cos A}{\sin A}$

**4.** (a) $\sin^2 A + \cos^2 A = 1$
(b) $\tan^2 A + 1 = \sec^2 A$
(c) $\cot^2 A + 1 = \csc^2 A$

**5.** (a) $\sin(90° \pm A) = \cos A$ $\quad$ $\csc(90° \pm A) = \sec A$
(b) $\cos(90° \pm A) = \mp \sin A$ $\quad$ $\sec(90° \pm A) = \mp \csc A$
(c) $\tan(90° \pm A) = \mp \cot A$ $\quad$ $\cot(90° \pm A) = \mp \tan A$
(d) $\sin(180° \pm A) = \mp \sin A$ $\quad$ $\csc(180° \pm A) = \mp \csc A$
(e) $\cos(180° \pm A) = -\cos A$ $\quad$ $\sec(180° \pm A) = -\sec A$
(f) $\tan(180° \pm A) = \pm \tan A$ $\quad$ $\cot(180° \pm A) = \pm \cot A$
(g) $\sin(-A) = -\sin A$; $\quad$ $\csc(-A) = -\csc A$
(h) $\cos(-A) = \cos A$; $\quad$ $\sec(-A) = \sec A$
(i) $\tan(-A) = -\tan A$; $\quad$ $\cot(-A) = -\cot A$

**6.** (a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
(c) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{\mp \tan A \tan B}$

(a) $\sin 2A = 2 \sin A \cos A$
(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$
(c) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

**8.** (a) $\sin \dfrac{A}{2} = \pm \sqrt{\dfrac{1 - \cos A}{2}}$

(b) $\cos \dfrac{A}{2} = \pm \sqrt{\dfrac{1 + \cos A}{2}}$

(c) $\tan \dfrac{A}{2} = \pm \sqrt{\dfrac{1 - \cos A}{1 + \cos A}} = \dfrac{\sin A}{1 + \cos A} = \dfrac{1 - \cos A}{\sin A}$

**9.** (a) $\sin A + \sin B = 2 \sin \dfrac{A + B}{2} \cos \dfrac{A - B}{2}$

(b) $\sin A - \sin B = 2 \cos \dfrac{A + B}{2} \sin \dfrac{A - B}{2}$

(c) $\cos A + \cos B = 2 \cos \dfrac{A + B}{2} \cos \dfrac{A - B}{2}$

(d) $\cos A - \cos B = - 2 \sin \dfrac{A + B}{2} \sin \dfrac{A - B}{2}$

**10.** (a) $\sin A \cos B = \dfrac{\sin (A + B) + \sin (A - B)}{2}$

(b) $\cos A \sin B = \dfrac{\sin (A + B) - \sin (A - B)}{2}$

(c) $\cos A \cos B = \dfrac{\cos (A + B) + \cos (A - B)}{2}$

(d) $\sin A \sin B = \cos (A - B) - \cos (A + B)$

For any $\triangle ABC$:

**11.** (a) $K = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A \quad (K = \text{area})$

(b) $K = \sqrt{s (s - a)(s - b)(s - c)}$, where $s = \frac{1}{2}(a + b + c)$

**12.** (a) Inradius: $r = \sqrt{\dfrac{(s - a)(s - b)(s - c)}{s}}$; $s = \frac{1}{2}(a + b + c)$

(b) Circumradius: $2R = \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

**13.** Law of Sines: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

**14.** Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

**15.** Law of Tangents: $\tan \dfrac{A - B}{2} = \dfrac{a - b}{a + b} \tan A + B$

**16.** Half-angle Formulas: $\tan \dfrac{A}{2} \quad s - a$

$\tan \dfrac{B}{2} = \dfrac{r}{s - b}$

$\tan \dfrac{C}{2} = \dfrac{r}{s - c}$, where $r$ = inradius.

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